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VERIFICATION OF ANALYTICAL ANTIDERIVATIVES FORMS USING CORRELATION ANALYSIS FOR MECHANICAL PROBLEMS

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Аналітичний пошук первісних функцій (невизначених інтегралів) широко використовується в математичному моделюванні різноманітних технічних, економічних, екологічних, біологічних, соціальних та інших процесів. У свою чергу, в задачах механіки є значний клас підзадач, при розв'язанні яких використовуються аналітичні методи інтегрування. До цих задач також відноситься проблема розробки аналітичних моделей навігаційно-балістичного забезпечення та моделей теорії керування в галузі ракетно-космічної техніки. Перевагою цього підходу в математичному моделюванні є можливість швидкого аналізу стану динамічних систем на різних часових інтервалах без розрахунків всіх попередніх станів.

У свою чергу, для деяких класів функцій існує кілька різних варіантів пошуку первісних, у результаті чого існує кілька різних форм первісних, які важко перевірити класичним способом у стандартній формі. В основному це пов'язано з вибором різноманітних комбінацій методів інтегрування, які використовуються при розробці аналітичних моделей, зокрема в задачах прикладної механіки.

Враховуючи зазначені складнощі верифікації множини первісних функції, у роботі пропонується метод, заснований на використанні кореляційного аналізу для перевірки відповідності їх аналітичних форм. При цьому масиви значень кожної первісної форми функції у певних вузлових точках пропонується представити у вигляді набору випадкових величин. З огляду на це, процес верифікації пропонується провести за допомогою стандартного підходу, заснованого на кореляційному аналізі (із застосуванням коефіцієнту кореляції Пірсона). Ефективність методу показана на прикладі перевірки первісних раціональної функції з квадратним тричленом, який піднесено до квадрату, в знаменнику. Такий підхід дасть змогу перевірити адекватність знаходження *i*-го варіанту первісної функції множині наявних первісних цієї функції та адаптувати задачу до стандартного вигляду.

Ключові слова: *первісна, метод верифікації, кореляційний аналіз, аналітична модель, механіка, інтегрування.*

An analytical search for antiderivative functions (indefinite integrals) is widely used in the mathematical simulation of various engineering, economic, ecological, biological, social, and other processes. In their turn, mechanical problems have many subproblems whose solution involves analytical integration methods. Among these problems is the problem of development of analytical models for navigation and ballistics support and control theory models in space rocket engineering. The advantage of this approach to mathematical simulation is a fast analysis of the state of dynamic systems on different time intervals without calculating all previous states.

In their turn, for some classes of functions, antiderivatives may be found in several different ways, as a result of which there exist several different forms of antiderivatives that are hard to verify by the classical method in standard form. This is mainly due to the choice of various combinations of integration methods used in the development of analytical models, in particular in problems of applied mechanics.

Taking into consideration these difficulties in the verification of the set of antiderivative functions, this paper proposes a method to check their analytical forms for correspondence with the use of correlation analysis. In doing so, the arrays of the values of each antiderivative form at certain nodal points are represented as a set of random variables. With this in mind, it is suggested that the verification process be conducted with the use of the standard approach based on correlation analysis (using Pearson's correlation coefficient). The efficiency of the method is shown by the example of verifying the antiderivatives of the reciprocal of a squared quadratic trinomial. This approach will make it possible to check the adequacy of the *i*-th candidate antiderivative and to adapt the problem to the standard form.

Keywords: *antiderivative, verification method, correlation analysis, analytical model, mechanics, integration.*

Introduction. Mathematical modeling of dynamic systems and processes is a key stage in the study of various phenomena and laws of nature, technical objects, biological, economic, social systems, etc. Usually, such systems and processes are modeled using the theory of ordinary differential equations (ODEs) [1 – 3] (mechanical systems, economic systems, physical processes, biological systems, etc.), difference equations (an alternative to modeling systems using ODEs in case of determined dynamics of changes in system parameters with time) [4, 5], Fourier series (time-periodic systems) [6, 7], the theory of infinitesimal calculus (piecewise continuous functions in moving control systems) [8, 9] and the theory of functions of a complex variable (AC circuits) [10]. In turn, when using the above mathematical apparatus, the theory of integral and differential calculus is used to find the solution (finding the coefficients of Fourier series, analytical solution of ODEs, etc.) [11, 12].

In turn, in the tasks of the field of rocket and space technology, the study of dynamic systems is one of the key issues [13 – 16]. Thus, modeling of spacecraft orbital and angular motion is often modeled using the ODE theory. So, using computer modeling in dynamic's and ballistic's problems, numerical methods are often used: for ODEs (ODE systems) (Euler, Runge-Kutta, Adams-Bashfort, Adams-Multon, Everhart, etc. methods), for numerical integration (formulas of rectangles, trapezoids, Simpson, etc.) [14]. However, with large limits and a small integration step, numerical methods require a significant number of integration steps in a cycle, which can significantly overload computing systems. Taking this into account, it is most expedient to search for analytical or numeric-analytical solutions if this possible. The use analytical or numeric-analytical can reduce the load on computing systems, which in turn will increase their performance. In this case, the development of mathematical models is carried out taking into account the use of traditional analytical methods of integration and differentiation [11, 12].

Problem statement and algorithm description Analytical integration is realized by finding the antiderivatives of functions. In turn, antiderivative of the function $f(x)$ on a certain interval is called a function $F(x)$ if it is continuous, differentiable and satisfies the condition [17] on this interval:

$$\begin{aligned} &F'(x) = f(x), \text{ or to the equation} \\ &dF(x) = f(x)dx. \end{aligned} \tag{1}$$

Given this, the indefinite integral is a family of antiderivatives of a function that differs by a parameter: the constant C . This is written as follows [11, 12, 17]:

$$\int f(x)dx = F(x) + C \ . \tag{2}$$

In turn, there are a number of functions that have different forms of antiderivatives depending on the chosen integration method. The correctness of finding such antiderivatives is rather difficult to identify in the standard form (2).

So, if the values of each function will be presented as arrays of data the statistical methods of verification can be used. The technique of absolute and relative errors usage for data validation is presented in the papers [18, 19]. Using this methodology for verification of antiderivatives forms should be based on the constant values of relative error in determined nodal points. In turn, this methodology is most suitable for verifying two analytical solutions. However, the usage this methodology requires significant number of nodal points which complicates the verification procedure. This is especially necessary for the verification of the analytical forms of antiderivatives with the results obtained by numerical integration (where the integration error depends on the features of the numerical method itself). Also, according [18], the use of the technique of data verification by calculating the relative error has limits in estimation of functions with near-zero values. In turn, the use of correlation analysis allows to consider the behavior of a function on a certain interval, taking into account estimates of the relationship with the variables of the compared function. This greatly simplifies the verification process and avoids the difficulties that arise when using methods [18, 19].

Proceeding from this, the paper proposes a method for verifying the forms of antiderivatives of a function using correlation analysis. In this case, the values of each of the antiderivative functions at certain nodal points are represented by sets of random variables. Further, for the i -th antiderivative $F_i(x)$, which needs to be verified, the mathematical expectation $\mathrm{M}[F_i(x)]$ and standard deviation $\sigma[F_i(x)]$ are calculated. After that, it is calculated the value of the mathematical expectation $\mathrm{M}[F_j(x)]$ and the standard deviation $\sigma[F_j(x)]$ of the antiderivative function $F_j(x)$ (obtained numerically, analytically, or numerical-analytically), with which $F_i(x)$ is compared. Then, the mathematical expectation of a two-dimensional random variable $\mathrm{M}[F_i(x),F_j(x)]$ created by the values at the nodal points of the functions $F_i(x)$ and $F_j(x)$ is calculated. In turn, the number of nodal points of $F_i(x)$ and $F_j(x)$ must be strictly the same and for the same values of the input arguments x of these functions. After that, the correlation coefficient (Pearson's correlation coefficient) is calculated as follows [20, 21]:

$$
\begin{gathered}
r\left(F_i(x),F_j(x)\right)=\frac{\mathrm{M}\left[F_i(x),F_j(x)\right]-\mathrm{M}[F_i(x)]\cdot\mathrm{M}\left[F_j(x)\right]}{\sigma\left(F_i(x)\right)\cdot\sigma\left(F_j(x)\right)},\\
\mathrm{M}[F_i(x)]=\frac{\sum_{n=1}^{N}F_i(x_n)}{N},\quad \mathrm{M}\left[F_j(x)\right]=\frac{\sum_{n=1}^{N}F_j(x_n)}{N},\\
\sigma[F_i(x)]=\sqrt{\frac{\sum_{n=1}^{N}\left(F_i(x_n)-\mathrm{M}[F_i(x)]\right)^2}{N}},\\
\sigma\left[F_j(x)\right]=\sqrt{\frac{\sum_{n=1}^{N}\left(F_j(x_n)-\mathrm{M}\left[F_j(x)\right]\right)^2}{N}},\\
\mathrm{M}\left[F_i(x),F_j(x)\right]=\frac{\sum_{n=1}^{N}F_i(x_n)\cdot F_j(x_n)}{N},\\
x_a\le x\le x_b,
\end{gathered}
\tag{3}
$$

where N is the number of nodal points; x_a, x_b are the borders of the antiderivatives verification interval.

In turn, the accuracy of the method depends on the number of selected nodal points N in a given interval, as well as the length of the verification interval itself. Also, in the case of comparing two antiderivative functions that have an analytical form, the calculation of mathematical expectations and variances can be carried out as for continuous random variables in the following form [21]:

$$
\begin{gathered}
r\left(F_i(x),F_j(x)\right)=\frac{K\left[F_i(x),F_j(x)\right]}{\sigma\left(F_i(x)\right)\cdot\sigma\left(F_j(x)\right)},\\
\mathrm{M}\left[F_i(x)\right]=\int_{x_a}^{x_b}x_iF_i(x)dx_i,\quad \mathrm{M}\left[F_j(x)\right]=\int_{x_a}^{x_b}x_jF_j(x)dx_j,\\
\sigma\left[F_i(x)\right]=\sqrt{\int_{x_a}^{x_b}\left(x_i-\mathrm{M}\left[F_i(x)\right]\right)^2F_i(x)dx_i},\\
\sigma\left[F_j(x)\right]=\sqrt{\int_{x_a}^{x_b}\left(x_j-\mathrm{M}\left[F_j(x)\right]\right)^2F_j(x)dx_j},\\
K\left[F_i(x),F_j(x)\right]=\int_{x_a}^{x_b}\int_{x_a}^{x_b}\left(x_i-\mathrm{M}\left[F_i(x)\right]\right)\left(x_j-\mathrm{M}\left[F_j(x)\right]\right)F_{i,j}(x_i,x_j)dx_idx_j,\\
x_a\le x\le x_b,
\end{gathered}
\tag{4}
$$

where $K\left[F_i(x),F_j(x)\right]$ is covariance; $F_{i,j}(x_i,x_j)$ is distribution function of a two-dimensional random variable.

Lema 1. *Thus, taking into account (3) and (4), the necessary and sufficient condition for the antiderivatives full correspondence to each other is the equality of the correlation coefficient* $r\left(F_i(x),F_j(x)\right)=1$. *In this case, if the function values* $F_i(x)$ *will increase or decrease by a certain amount, then the function values* $F_j(x)$ *will increase or decrease by the same amount, which correspond to the definition (2). In turn, if the* $r\left(F_i(x),F_j(x)\right)<1$ *it can be concluded that obtained form of antiderivative corresponds to other forms of the function's anti-derivatives with a certain degree of confidence.*

Estimation of algorithm implementation. Let's consider a function $y(x)=\dfrac{1}{(x^2+a^2)^2}$. So $y(x)$ is a rational function whose denominator is a square trinomial (for $A=1, B=0, C=a^2$) raised to the second power. Functions of this type are often encountered in the description of automatic control systems for special ODE's right-hand sides whose inverse Laplace transform has the form $f(t)=\dfrac{1}{2\omega^3}\left(\sin\omega t-\omega t\cos\omega t\right)$, $\omega\ge 0$. Also, $y(x)$ can be included in the ODEs right-hand sides, whose inverse Laplace transform has the form $f(t)=t\cos\omega t$, $\omega\ge 0$. In turn, in some cases, it is necessary to directly integrate the Laplace s-domains (by applying the Laplace s-domains integration theorem). Such cases can be finding Laplace s-domains from functions of type $g(t)=\dfrac{f(t)}{t}$, when searching for an analytical solution for more convenient adaptation of mathematical expressions to machine language. This is often necessary when developing software without using application packages with built-in operational calculus libraries (such as Matlab, Scilab, MathCad and their analogues).

Thus, the first variant of the analytical finding of the antiderivative for $y(x)=\dfrac{1}{(x^2+a^2)^2}$ is the use of the well-known recursive formula [12] of the type:

$$I_{k+1}=\frac{1}{2ka^2}\frac{x}{(x^2+a^2)^k}+\frac{2k-1}{2ka^2}I_k, \quad k\in\mathbb{N}. \tag{5}$$

where $I_{k+1}=\int\dfrac{1}{(x^2+a^2)^{k+1}}dx$; $I_k=\int\dfrac{1}{(x^2+a^2)^k}dx$.

In this case $k=1$. Then, using formula (5), the expression for the function $y(x)=\dfrac{1}{(x^2+a^2)^2}$ first form antiderivative $F_1(x)$ can be written as follows:

$$\begin{aligned} F_1(x)&=\int\frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^2}\cdot\frac{x}{(x^2+a^2)}+\frac{1}{2a^2}\int\frac{dx}{x^2+a^2}= \\ &=\frac{1}{2a^3}\arctan\frac{x}{a}+\frac{1}{2a^2}\cdot\frac{x}{(x^2+a^2)}+C \end{aligned} \tag{6}$$

The second variant for finding the antiderivative for the function $y(x)=\dfrac{1}{(x^2+a^2)^2}$ is to use a trigonometric substitution of the form: $x=a\cdot\tan(z)$, $z=\arctan\left(\dfrac{x}{a}\right)$, $dx=\dfrac{a}{\cos^2(z)}dz$. Using this substitution, the indefinite integral of the function $y(x)$ takes the following form:

$$F_2(x)=\int\frac{adz}{\cos^2(z)\cdot\left(a^2\tan^2(z)+a^2\right)^2},\quad z=\arctan\left(\frac{x}{a}\right)$$

$$\begin{aligned} F_2(x)&=\int\frac{adz}{\cos^2(z)\cdot a^4\,\dfrac{1}{\cos^4(z)}}= \\ &=\frac{1}{a^3}\int\cos^2(z)dz=\frac{1}{a^3}\int\frac{1+\cos(2z)}{2}dz= \\ &=\frac{1}{2a^3}\left(z+\frac{\sin(2z)}{2}\right)=\frac{1}{2a^3}\arctan\left(\frac{x}{a}\right)+\frac{\sin\left[2\arctan\left(\dfrac{x}{a}\right)\right]}{4a^3}+C\ . \end{aligned} \tag{7}$$

Based on the obtained values of the variants of the antiderivatives $F_1(x)$ and $F_2(x)$ for the function $y(x)$, it can be seen that the values of the second terms have different representations. In turn, according to the properties of the universal trigonometric substitution [12], the expression $\dfrac{\sin\left[2\arctan\left(\dfrac{x}{a}\right)\right]}{4a^3}$ can be reduced to $\frac{1}{2a^2}\cdot\frac{x}{(x^2+a^2)}$. In this case, it can be seen that the values of the antiderivatives $F_1(x)$ and $F_2(x)$ are interconnected by the relations of the universal trigonometric substitution when using the integration methods (6), (7) and the use of correlation analysis for verification is optional.

In turn, if apply the method of two substitutions, it can be obtained another form of the antiderivative for functions of the type $y(x)$. So, when use the substitution 1 for the function $y(x)$: $x^2+a^2=t$, $x=\pm\sqrt{t-a^2}$, $dx=\frac{\pm dt}{2\sqrt{t-a^2}}$,

finding of the antiderivative is reduced to the form:

$$F_3(x)=\pm\frac{1}{2}\int\frac{dt}{t^2\sqrt{t-a^2}},\quad t=x^2+a^2. \tag{8}$$

Further, use the substitution 2: $t=\frac{1}{m}$, $m=\frac{1}{t}$, $dt=-\frac{dm}{m^2}$ expression (8) is reduced to the form:

$$F_3(x)=\mp\frac{1}{2}\int\frac{\sqrt{m}dm}{\sqrt{1-ma^2}},\quad m=\frac{1}{x^2+a^2}. \tag{9}$$

Multiplying the numerator and denominator by $\sqrt{m}$, expression (9) can be written as follows:

$$F_3(x)=\mp\frac{1}{2}\int\frac{mdm}{\sqrt{m-m^2a^2}},\quad m=\frac{1}{x^2+a^2}.$$

The antiderivative of this expression has the following form:

$$F_3(x)=\mp\frac{1}{2}\int\frac{mdm}{\sqrt{m-m^2a^2}}=$$
$$=\pm\frac{1}{2a^2}\sqrt{m-m^2a^2}\mp\frac{1}{4a^3}\arcsin\left(2ma^2-1\right)+C,\quad m=\frac{1}{x^2+a^2},$$

or in final form:

$$\begin{aligned}&F_3(x)=\frac{1}{2a^2}\frac{x}{x^2+a^2}\mp\frac{1}{4a^3}\arcsin\left(\frac{a^2-x^2}{a^2+x^2}\right)+C,\\&-,\text{ if }\quad x>0,\\&+,\text{ if }\quad x<0,\\&C=\begin{cases}C_1,\text{ if } & x>0\\C_2,\text{ if } & x<0.\end{cases}\end{aligned}, \tag{10}$$

where C_1 and C_2 are the constants of integration on first and second intervals depend of sign.

It can be seen from the obtained result that the form of the antiderivative function $F_3(x)$ has significant differences in the second term, both in form and in sign, from the second $F_1(x)$ and $F_2(x)$ terms. In turn, verification by analytical methods can be quite cumbersome and lead to additional errors. Thus, in this case, it is advisable to apply a verification method based on correlation analysis.

Based on the ***Lema 1***, let's determine the verification conditions for the antiderivatives $F_1(x)$, $F_2(x)$ and $F_3(x)$ of the considered function $y(x)$ on a certain interval $[x_1, x_2]$ in the following form:

$$\begin{aligned} &r(F_1(x), F_2(x)) = 1, \\ &r(F_1(x), F_3(x)) = 1, \\ &r(F_2(x), F_3(x)) = 1, \\ &x_1 \le x \le x_2. \end{aligned} \qquad (11)$$

For a numerical experiment, let's take for example the a value of function $y(x) = \dfrac{1}{(x^2 + a^2)^2}$ equal to 10. Then, the values of antiderivatives $F_1(x)$, $F_2(x)$ and $F_3(x)$ will be have the next form:

$$\begin{aligned} F_1(x) &= \frac{1}{200} \cdot \frac{x}{(x^2 + 100)} + \frac{1}{2000} \arctan\left(\frac{x}{10}\right) + C, \\ F_2(x) &= \frac{1}{2000} \arctan\left(\frac{x}{10}\right) + \frac{\sin\left[2 \arctan\left(\frac{x}{10}\right)\right]}{4000} + C, \\ F_3(x) &= \frac{1}{200} \cdot \frac{x}{\left(x^2 + 100\right)} \mp \frac{1}{4000} \arcsin\left(\frac{100 - x^2}{100 + x^2}\right) + C. \end{aligned}$$

Let's set the constants $C = 0$ for $F_1(x)$, $F_2(x)$ and $C_1 = 0.0007854; C_2 = 0$ for $F_3(x)$. Then using the open-source package of mathematical applications SciLab carry out the verification of these antiderivatives. To do this, let's take a breakdown interval $[x_1, x_2]$ from -50 to 50 with a step of 0.001 and build graphs of all three antiderivatives (Fig. 1) at $a = 10$.

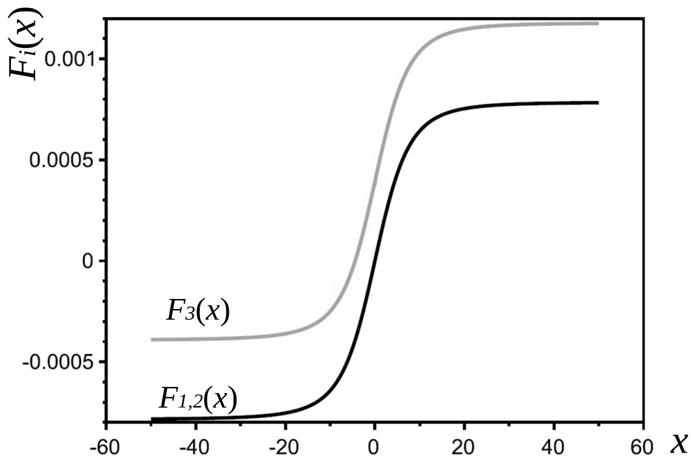

Fig. 1 – The graphs of antiderivatives $F_1(x)$, $F_2(x)$ and $F_3(x)$ of the function $y(x)$ on the interval $x = [-50, 50]$ at $a = 10$

It has been determined that if C_1 and C_2 of the antiderivative function $F_3(x)$ have wrong values, $F_3(x)$ suffers a discontinuity of the first kind (jumps) at the point $x = 0$, which does not satisfy condition (1). In this case, the function $F_3(x)$ cannot be fully called the antiderivative of $y(x)$. In turn, if $C_1 = 0.0007854$ and $C_2 = 0$, the correlation coefficients are equal $r\left(F_1(x), F_3(x)\right) = r\left(F_2(x), F_3(x)\right) = 1.0$, which satisfies condition (11) (fig.1).

Thus, the proposed verification method made it possible to fully assess the correspondence of the forms of the antiderivatives of the function $y(x)$ to their direct definition (1). Also, if function has any different constants depending on the interval as $F_3(x)$, the correlation methodology helps to determine these constants correctly. It is also established that when applying various integration methods, the properties of the antiderivative may change.

Discussion. The use of the modern computer technology power in the application of the proposed method makes it possible to quickly verify the analytically found form of the antiderivative function to its other forms. In turn, if there are less than two such forms, then it is advisable to carry out verification with the expansion of the function in a Maclaurin (Taylor) series on a given interval and with integration values using numerical methods. The approach of representing functions describing perturbative influences by the series is especially often used in the analysis of spacecraft dynamics [15, 16, 20]. In this case, taking into account the calculation error when using one or another numerical method, as well as the degree of expansion of the original function in the Maclaurin series, the application of the proposed verification method may not give an absolute correlation even in the absence of calculation errors. So, for example, let's expand the function $y(x) = \dfrac{1}{(x^2+100)^2}$ in a Maclaurin series up to the 8th degree and integrate. The expansion has the following form:

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \ldots + \frac{y^{(7)}(0)}{7!}x^7 + \frac{y^{(8)}(0)}{8!}x^8 = \\ = \frac{1}{100^2} - \frac{2}{100^3}x^2 + \frac{72}{24\cdot 100^4}x^4 - \frac{2880}{720\cdot 100^5}x^6 + \frac{201600}{40320\cdot 100^6}x^8 .$$

By directly integrating, it will be got:

$$Y(x) = \frac{1}{100^2}x - \frac{2}{3\cdot 100^3}x^3 + \frac{72}{120\cdot 100^4}x^5 - \frac{2880}{5040\cdot 100^5}x^7 + \frac{201600}{9\cdot 40320\cdot 100^6}x^9 .$$

Let's plot graphs (Fig. 2) and carry out a correlation analysis of the antiderivatives $F_1(x)$, $F_2(x)$, and the integral of the Maclaurin expansion $Y(x)$ of the function $y(x) = \dfrac{1}{(x^2+100)^2}$ on the interval $x = [-10, 10]$.

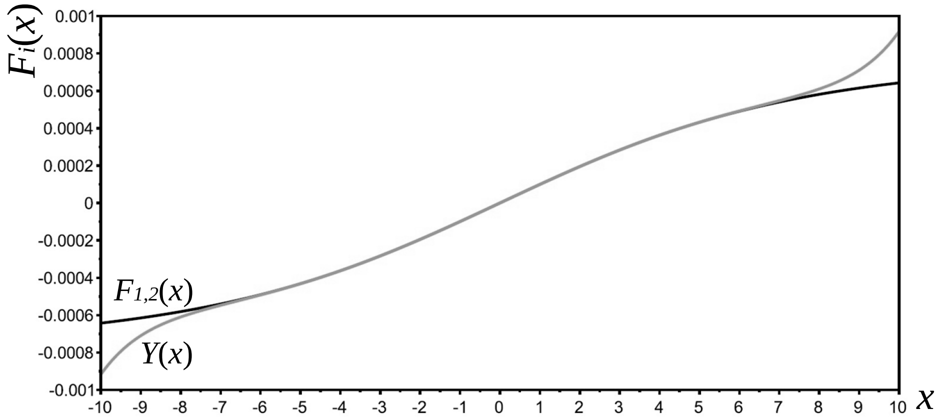

Fig. 2 – The graphs of antiderivatives $F_1(x)$, $F_2(x)$ and the integral of the Maclaurin expansion $Y(x)$ of the function $y(x)$ on the interval $x=[-10,10]$ at $a=10$

Thus, when using correlation analysis, it has been obtained $r\left(F_1(x),Y(x)\right)=r\left(F_2(x),Y(x)\right)=0.9947596$. In turn, if the degree of expansion in the Maclaurin series is increased to 14, the correlation will be **0.9951848**. In such cases, when using the proposed verification method, it is necessary to set the confidence value of the correlation coefficient (according to Lemma 1), depending on the required calculation accuracy, in which the found analytical value of the antiderivative can be considered correct. This approach can be used to search for analytical solutions to differential equations and their systems, the right-hand sides of which are difficult functions. In turn, the chosen confidence value of the correlation coefficient will affect the accuracy of calculations when using the found analytical solution.

Conclusions. A method for verifying the analytical search for antiderivatives using correlation analysis is proposed. On the example of finding antiderivatives of a function $y(x)=\dfrac{1}{(x^2+a^2)^2}$ and their further verification it has been shown the example of proposed method implementation. A numerical experiment using the proposed verification method showed different properties of antiderivatives depending on the chosen integration method for the function $y(x)=\dfrac{1}{(x^2+a^2)^2}$.

Taking this into account, it is expedient to apply the proposed method of verification of the analytical search for antiderivatives to difficult functions that have different forms of antiderivatives. Also, this method can be used to analyze the confidence values of the accuracy of the found numerical, numeric-analytical and analytical solutions for integrating various difficult functions, ODEs which are often uses in different areas in mechanics.

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