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# MATHEMATICAL MODELING OF ARTICULATED PASSENGER TRAIN SPATIAL VIBRATIONS

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Проблема розвитку швидкісного залізничного транспорту є актуальною для України. У багатьох країнах для цього використовують зчленовані поїзди. Оскільки зв'язки між вагонами в такому поїзді різняться між собою, для дослідження його динамічних характеристик необхідно складати повну модель коливань поїзда, а не окремого вагона. Стаття присвячена розробці математичної моделі просторових коливань зчленованого пасажирського поїзда. Даний поїзд складається з 7-ми вагонів: один моторний вагон, один перехідний вагон, три зчленовані вагони, далі ще один перехідний вагон і знову один моторний вагон. Диференціальні рівняння руху поїзда по колії довільної форми складено як рівняння Лагранжа другого роду. Прийнято до уваги всі необхідні особливості конструкції екіпажів. Зчленовані вагони мають спільні візки з суміжними вагонами та з перехідним вагоном і зв'язок між вагонами забезпечується за допомогою шарніра. Робота шарніра між двома вагонами моделюється пружинами та демпферами, що діють в горизонтальному та вертикальному напрямах. Демпфери між кузовами двох сусідніх вагонів моделюються як демпфери в'язкого тертя. Складено систему з 257 диференціальних рівнянь другого порядку, які описують рух зчленованого поїзда вздовж прямолінійної, криволінійної та перехідної ділянок колії з урахуванням випадкових нерівностей колії. На основі отриманої математичної моделі розроблено алгоритм та відповідне програмне забезпечення для моделювання широкого кола випадків, що включають всі можливі комбінації параметрів для елементів поїзда та технічного стану колії. Дослідження власних коливань поїзда показало його стабільний рух у всьому діапазоні розглянутих швидкостей (40 км/год – 180 км/год). Результати, отримані для руху поїзда вздовж колії, призначеної для швидкісного руху, показали, що всі динамічні характеристики та показник якості руху забезпечують безпеку руху поїзда та комфортні умови для пасажирів.

***Ключові слова:*** *математична модель, зчленований поїзд, просторові коливання, динамічні характеристики.*

The problem of high-speed railway transport development is important for Ukraine. In many countries articulated trains are used for this purpose. As the connections between cars in such a train differ from each other, to investigate its dynamic characteristics not a separate car, but a full train vibrations model is necessary. The article is devoted to the development of the mathematical model for articulated passenger train spatial vibrations. The considered train consists of 7 cars: one motor-car, one transitional car, three articulated cars, one more transitional car and again one motor-car. Differential equations of the train motion along the track of arbitrary shape are set in the form of Lagrange's equations of the second kind. All the necessary design features of the vehicles are taken into account. Articulated cars have common bogies with adjoining cars and a transfer car and the cars are united by the hinge. The operation of the central hinge between two cars is modeled using springs and dampers acting in the horizontal and vertical directions. Four dampers between two adjacent car-bodies act as dampers for pitching and hunting and are represented in the model by viscous damping. The system of 257 differential equations of the second order is set, which describes the articulated train motion along straight, curved, and transitional track segments with taking into account random track irregularities. On the basis of the obtained mathematical model the algorithm and computational software has been developed to simulate a wide range of cases including all possible combinations of parameters for the train elements and track technical state. The study of the train self-exited vibrations has shown the stable motion in all the range of the considered speeds (40 km/h – 180 km/h). The results obtained at the train motion along the track maintained for the speedy motion have shown that all the dynamic characteristics and ride quality index insure train safe motion and comfortable conditions for the travelling passengers.

***Keywords:*** *mathematical model, articulated train, spatial vibrations, dynamic characteristics*.

**Introduction.** The successful operation of high-speed passenger railway transport in Europe shows the need to develop a similar railway network in Ukraine. In different countries issues related to the creation of high-speed rolling stock are resolved in different ways. One of the principles of high-speed trains creating is the principle of articulation, which involves the support of adjacent cars on common bogies located between them [1 – 3]. For an articulated train the key element is the coupling design, which ensures the stability of two series cars.

A typical articulated train consists of a locomotive followed by one transfer car, several articulated cars, one more transfer car, and another locomotive. The transfer car has an independent bogie on the locomotive side and a common articulated bogie with an adjacent passenger car. The dynamic interaction between cars in such a train cannot be estimated considering a separate car. Therefore, to model the vibrations of articulated trains and assess their dynamic qualities, it is necessary to consider a model consisting of several vehicles [4].

**Mathematical model.** The study of railway vehicles dynamic characteristics is associated with the consideration of mechanical systems with many degrees of freedom [5]. The reliability of the results obtained is determined, first of all, by the correct choice of the calculation scheme of the train under consideration. Therefore, the developed design scheme of an articulated passenger train should take into account the design features and characteristics of the cars load-bearing elements joints as fully as possible. The high-speed train under consideration is shown on Fig. 1. The transfer car has an independent bogie on the locomotive side and a common articulated bogie with an adjacent car. The independent bogies of the transfer cars are connected to the body in the usual way. Articulated cars share bogies with adjoining cars and a transfer car. The bogies of the articulated cars have two-stage suspension. The first stage has an elastic element working in longitudinal, horizontal lateral and vertical directions and damping elements. The second stage of suspension includes the connections between the bogie frame and the car-body and the connections between the frame and the articulation pivot. The operation of the central hinge between two cars is modeled using springs and dampers acting in the horizontal and vertical directions. Four dampers between two adjacent car-bodies act as dampers for pitching and hunting and are represented in the model by viscous damping.

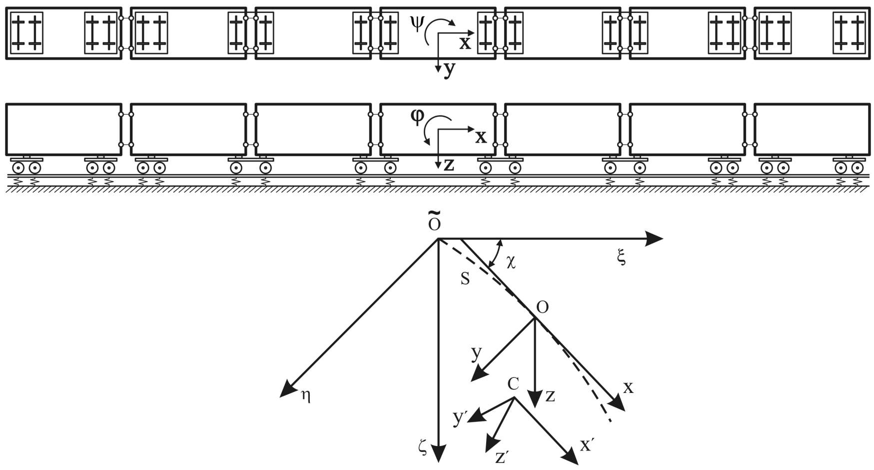


Fig. 1

A standard motor car of an electric train is considered as a locomotive. Thus, the train under consideration consists of 7 car-bodies, 4 bolsters, 10 bogies and 20 wheelsets. The entire composition of such a train can be modeled as a system of 41 rigid bodies with 246 degrees of freedom as a whole.

We consider the motion of an articulated passenger train along an elastic-viscous-inertial track, which is modeled by a mass reduced to each wheel (forty reduced masses), which has only vertical and horizontal lateral displacements and rests in these directions on springs and viscous dampers that simulate elastic-dissipative properties of rails and subrail base. Thus, in the general case, the system has 41 × 6 + 40 × 2 = 326 degrees of freedom.

A fixed coordinate system $\tilde{O}\xi\eta\zeta$ with its origin at the track centre line on a rail top level is chosen for describing the motion of the train along track sections of arbitrary shape, and for each rigid body two moving coordinate systems are chosen: a natural one $Oxyz$ and associated with rigid body $Cx'y'z'$ ($Cx'$, $Cy'$, $Cz'$ are the principal central axes of inertia). All of the coordinate systems are right, and the axes $\tilde{O}\xi$, $Ox$, $Cx'$ are directed from left to right, and the axes $\tilde{O}\zeta$, $Oz$, $Cz'$ are directed down (Fig. 1) [6].

The axes of the natural coordinate system are directed along the tangent, normal and binormal to the track axis, respectively. The origin of coordinates $O$ for each rigid body is at a distance $s$ from its position at the initial time (here $s$ is the distance travelled). The position of the natural system of coordinates relative to the stationary one is characterized by the arc coordinate $s$ along the track, by the angle $\chi$ between the axes $Ox$ and $\tilde{O}\xi$ in the plane, and the angles $\varphi_h$ and $\theta_h$ between these axes in vertical planes, which were determined by the elevation of the outer rail $h_r$ in curve. Parameters of the rail $\chi, h_r$ are the given functions of coordinate $s$.

When referring to the coordinate system of bodies the following subscripts are used: the car-body – $f_i$ $(i=\overline{1,7})$, the bolster – $b_i$ $(i=\overline{1,4})$, the frame – $sl$ $(l=\overline{1,10})$, the wheelset – $i$ $(i=\overline{1,20})$ is the number of a wheelset in the direction of motion), the wheel – $ij$ ($j$ = 1 for the car right side, $j$ = 2 for the car left side), rails at the points of contact – $rij$ .

Displacements $x, y, z$ and rotation angles $\psi, \varphi, \theta$ of separate bodies describe longitudinal, horizontal lateral, vertical displacements, hunting, pitching, rolling of a rigid body respectively. Positive directions of displacements are shown in Fig. 1 by arrows.

To determine the number of degrees of freedom for the mechanical system considered, the constraints imposed on the bodies' displacements as the generally accepted assumptions and design features of the train cars are taken into account.

Wheelsets' vertical displacements and rolling are expressed in terms of the vertical rail track irregularities:

$$z_i = \frac{z_{i1} + z_{i2}}{2}; \quad \theta_i = \frac{z_{i2} - z_{i1}}{2d_1}; \ (i = \overline{1,20}), \tag{1}$$

where $z_{ij}$ is the vertical displacement of the $i$-th wheelset $j$-th wheel, which depends on the rail irregularities and rail vertical displacement; $2d_1$ is the distance between the wheelset wheels' mean rolling radii.

It is assumed that the radii $r$ of the wheels' mean rolling circles are equal. Then at the coincidence of the track and bogie longitudinal planes of symmetry all the wheels turn through the same angle:

$$\varphi_i = -\frac{x_i + s}{r}, \ (i = \overline{1,20}) . \tag{2}$$

Bolsters of the motor cars are moving together with the car-bodies in the following directions:

$$
\begin{aligned}
&z_{bk} = z_f + (-1)^k l\varphi_f; \\
&\varphi_{bk} = \varphi_f; \\
&\theta_{bk} = \theta_f \,(k = 1,4),
\end{aligned}
\tag{3}
$$

where $2l$ is the motor car base.

Because of the rigidity of the longitudinal rods bolsters angular displacements in the horizontal plane are equal to the corresponding displacements of the bogies' frames:

$$\psi_{bk} = \psi_{sl}; \ (k = 1,4;\, l = 1,2). \tag{4}$$

With constrains (1) – (4) the "articulated train-track" system has 326-76=250 degrees of freedom.

To investigate train motion at the transitional modes it is necessary to add one more coordinate for each car. It corresponds to the change of the car absolute displacement in the longitudinal direction – $s_i \ (i = \overline{1,7})$. So the number of degrees of freedom for the train will be equal to 257.

Differential equations of the train motion along the track of arbitrary shape are set in the form of Lagrange's equations of the second kind [7]:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial \Pi}{\partial q_i} + \frac{\partial \Phi}{\partial \dot{q}_i} = Q_i + S_i^*, \quad (i = \overline{1,257}), \tag{5}$$

where $q_i, \dot{q}_i$ are the generalized coordinates and their velocities; $T$ is the kinetic energy; $\Pi$ is the potential energy; $\Phi$ is the dissipative function; $Q_i$ are the generalized non-potential forces; $S_i^*$ are the applied external forces.

For each rigid body the kinetic energy is determined by the Koenig theorem. In general, the expression for the kinetic energy of the $i$-th rigid body can be written as follows:

$$
\begin{aligned}
T_i = {}& \frac{1}{2} m_i \left[ (\dot{s} + \dot{x}_i - y_i \dot{\chi}_i)^2 + (\dot{y}_i + x_i \dot{\chi}_i)^2 + (\dot{z}_i - \dot{h}_{ri})^2 \right] + \\
& \frac{1}{2} I_{xi} (\dot{\theta}_i + \dot{\theta}_{hi})^2 + \frac{1}{2} I_{zi} (\dot{\psi}_i + \dot{\chi}_i)^2 + \frac{1}{2} I_{yi} (\dot{\varphi}_i + \dot{\varphi}_{hi})^2,
\end{aligned}
$$

where $m_i$ is the mass of the $i$-th rigid body; $I_i$ with appropriate subscripts denote the principal central moments of inertia of the $i$-th body; $\dot{\chi}_i = vK_i$, $K_i$ is the track curvature under the $i$-th body; $v$ is the speed of motion; $h_{ri}$ is the track elevation under the $i$-th body mass centre caused by the outer rail elevation in the curve $h_r = \theta_h 2d_1$.

In accordance with the accepted assumptions the kinetic energy of the system modeling the track can be written as follows:

$$T_t = \frac{1}{2} m_{rh} \sum_{i=1}^{20} \sum_{j=1}^{2} \dot{y}_{rij}^2 + \frac{1}{2} m_{rv} \sum_{i=1}^{20} \sum_{j=1}^{2} \dot{z}_{rij}^2 \,,$$

where $m_{rh}, m_{rv}$ are the track masses reduced to one wheel in the horizontal lateral and vertical directions, respectively.

The potential energy of the considered system ($\Pi$) is defined as the sum of elastic deformations energy ($\Pi_1$) and the energy changes as a result of rising or lowering the system bodies' mass centres ($\Pi_2$).

The potential energy of the system $\Pi_1$ is defined by the Clapeyron theorem as the sum of energies accumulated in the elastic elements of the system during their deformation and has the form:

$$\Pi_1 = \frac{1}{2}\left[\begin{array}{l} \sum_{i=1}^{20}(k_{czi}\Delta_{czi}^2 + k_{cyi}\Delta_{cyi}^2 + k_{cxi}\Delta_{cxi}^2) + \sum_{i=1}^{40}(k_{azi}\Delta_{azi}^2 + k_{ayi}\Delta_{ayi}^2 + k_{axi}\Delta_{axi}^2) \\ + \sum_{i=1}^{16}(k_{syi}\Delta_{syi}^2 + k_{sxi}\Delta_{sxi}^2) + \sum_{i=1}^{4}(k_{shz}\Delta_{shzi}^2 + k_{shy}\Delta_{shyi}^2 + k_{shyx}\Delta_{shxi}^2) \\ + k_\theta \sum_{i=1}^{4}\Delta_{\theta i}^2 + \sum_{i=1}^{8}(k_{skyi}\Delta_{skyi}^2 + k_{skxi}\Delta_{skxi}^2) + \sum_{i=1}^{20}\sum_{j=1}^{2}(k_{rz}\Delta_{rzij}^2 + k_{ry}\Delta_{ryij}^2) \end{array}\right],$$

where $k_{czi}, k_{cyi}, k_{cxi}$ are the stiffnesses of the $i$-th elastic element of the bogie central suspension in the vertical, horizontal lateral and longitudinal directions; $\Delta_{czi}, \Delta_{cyi}, \Delta_{cxi}$ are the deflections of the $i$-th elastic element of the central suspension in the vertical, horizontal lateral and longitudinal directions; $k_{azi}, k_{ayi}, k_{axi}$ are the stiffnesses of the elastic elements installed in the axle box above the $i$-th wheelset in the vertical, horizontal lateral and longitudinal directions; $\Delta_{azi}, \Delta_{ayi}, \Delta_{axi}$ are the deflections of the elastic elements installed in the axle box above the $i$-th wheelset in the vertical, horizontal lateral and longitudinal directions; $k_{syi}, k_{sxi}$ are the stiffnesses of the bogie axle box/frame additional connections in the horizontal lateral and longitudinal directions; $\Delta_{syi}, \Delta_{sxi}$ are the deflections of the $i$-th axle box rod in horizontal lateral and longitudinal directions; $k_{shz}, k_{shy}, k_{shx}$ are the hinge stiffnesses in the vertical, horizontal lateral and longitudinal directions; $\Delta_{shzi}, \Delta_{shyi}, \Delta_{shxi}$ are the hinge deflections in the vertical, horizontal lateral and longitudinal directions; $k_\theta$ is the torsion bar twist stiffness; $\Delta_{\theta i}$ is the angle deflection at the car-body and the $i$-th bogie rolling; $k_{skyi}, k_{skxi}$ are the stiffnesses of the rubber dampers in connection of car-body and bogie bolster in horizontal lateral and longitudinal directions; $\Delta_{skyi}, \Delta_{skxi}$ are the corresponding deflections; $k_{rz}, k_{ry}$ are the vertical and horizontal lateral stiffnesses of the track; $\Delta_{rzij}, \Delta_{ryij}$ are the vertical and horizontal lateral deflections of the track under the $j$-th wheel of the $i$-th wheelset.

Mutual bodies' displacements leading to elastic element deformations for the central and axle box suspensions are determined in usual way [5]. The displacements of the bogie with central suspension at one side and the hinge at the other side (articulated bogie) can be written in follows:

– vertical displacements of springs and hinge elastic elements:

$$\Delta_{z1} = z_{k1} + l_1\varphi_{k1} - z_r + a\varphi_r;$$
$$\Delta_{z2} = z_{k2} - l_2\varphi_{k2} - z_r - a^*\varphi_r - b(\theta_{k2} + \theta_{h2} - \theta_r - \theta_{hr});$$
$$\Delta_{z3} = z_{k2} - l_2\varphi_{k2} - z_r - a^*\varphi_r + b(\theta_{k2} + \theta_{h2} - \theta_r - \theta_{hr});$$

– horizontal lateral displacements of springs and hinge elastic elements:

$$\Delta_{y1} = y_{k1} - l_1(\psi_{k1} + \chi_{k1}) - y_r - a(\psi_r + \chi_r) - u_1;$$
$$\Delta_{y2} = \Delta_{y3} = y_{k2} + l_2(\psi_{k2} + \chi_{k2}) - y_r + a^*(\psi_r + \chi_r) - u_2;$$

– longitudinal displacements of springs and hinge elastic elements:

$$\Delta_{x1} = x_{k1} - x_r;$$
$$\Delta_{x2} = x_{k2} - x_r + b(\psi_{k2} + \chi_{k2} - \psi_r - \chi_r);$$
$$\Delta_{x3} = x_{k2} - x_r - b(\psi_{k2} + \chi_{k2} - \psi_r - \chi_r);$$

where $l_1$ is the distance from the transitional car mass centre to the hinge in longitudinal direction; $a$ is the distance from the frame mass centre to the hinge; $l_2$ is the distance from the articulated car mass centre to the central suspension element; $a^*$ is the distance from the frame mass centre to the central suspension element; $b$ is the distance between central suspension elements in lateral direction; $u_i \approx \frac{1}{2} l_i^2 K$ is the arch rise of the curvilinear track (in horizontal plane) under the car-body mass centre in the limit of car base ( $K_f$ is the track curvature under the car mass centre);

The potential energy caused by rising or lowering the $i$-th body centre of gravity with taking into account the curvilinear motion is defined as follows:

$$\Pi_2 = -\sum_{i=1}^{41} m_i g(\theta_{hi} y_i + z_i),$$

where $g$ is the acceleration due to gravity; $\theta_{hi}$ is an angle between the horizontal plain under the $i$-th body mass center and the track plate because of the outer rail elevation on the inner one.

The constructed calculation model takes into account the effect of viscous forces in the vertical and horizontal deflections of the suspension.

Dissipation function for the considered system has the form:

$$\Phi = \frac{1}{2}\left[\begin{array}{l} \sum_{i=1}^{20}\beta_{czi}\dot{\Delta}_{czi}^2 + \sum_{i=1}^{40}\beta_{azi}\dot{\Delta}_{azi}^2 + \sum_{i=1}^{4}\beta_{ci}\dot{\Delta}_{ci}^2 + \sum_{i=1}^{4}(\beta_{shzi}\dot{\Delta}_{shzi}^2 + \beta_{shyi}\dot{\Delta}_{shyi}^2 + \\ + \beta_{shxi}\dot{\Delta}_{shxi}^2) + \sum_{1}^{16}\beta_{xi}\dot{\Delta}_{xi}^2 + \sum_{i=1}^{20}\sum_{j=1}^{2}(\beta_{rz}\dot{\Delta}_{rzij}^2 + \beta_{ry}\dot{\Delta}_{ryij}^2) \end{array}\right]$$

where $\beta_{czi}$ is the coefficient of energy dissipation for the $i$-th damper of the central suspension in the vertical direction; $\beta_{azi}$ is the coefficient of energy dissipation in the elastic elements installed in the axlebox in the vertical direction; $\beta_{ci}$ is the coefficient of energy dissipation of the deviating hydrodamper between

$i$-th bogie bolster and frame; $\beta_{xi}$ is the coefficient of energy dissipation for the $i$-th damper installed between the car-bodies; $\beta_{rz}, \beta_{ry}$ are the coefficients of the energy dissipation in the track in the vertical and horizontal lateral directions; $\dot{\Delta}$ are the relative velocities of bodies connected by viscous dissipative elements.

Generalized forces $Q_i$ are defined as coefficients of variations of generalized coordinates in expressions of creep forces $T_{xij}$, $T_{\alpha ij}$ possible work [8]. In determining the forces acting on the wheel in the horizontal lateral direction, components of the force of gravity are taken into account (in addition to the creep forces). Contact point coordinates on the surfaces of the wheel and the corresponding rail are determined in accordance with the theory described in the article [9].

After the expressions of kinetic and potential energy, dissipation function, generalized and external forces are put in (5), a system of nonlinear differential equations of the 514-th order is obtained, which describe the articulated train motion along straight, curved, and transitional track segments with taking into account random track irregularities.

On the basis of the obtained mathematical model the algorithm and computational software has been developed to simulate a wide range of cases including all possible combinations of parameters for the train elements and track technical state. Calculated estimation of train dynamic indices has been done by the solution of non-linear differential equations described above. Non-linear differential equations have been solved by the Adams-Bashfort method.

**Results of calculations.** The study of the train self-exited vibrations has shown the stable motion in all the range of the considered speeds (40 km/h – 180 km/h). Time histories of the wheels hunting at the train motion along the straight track without irregularities for the speeds of 40 km/h, 100 km/h and 180 km/h are given on Fig. 2.

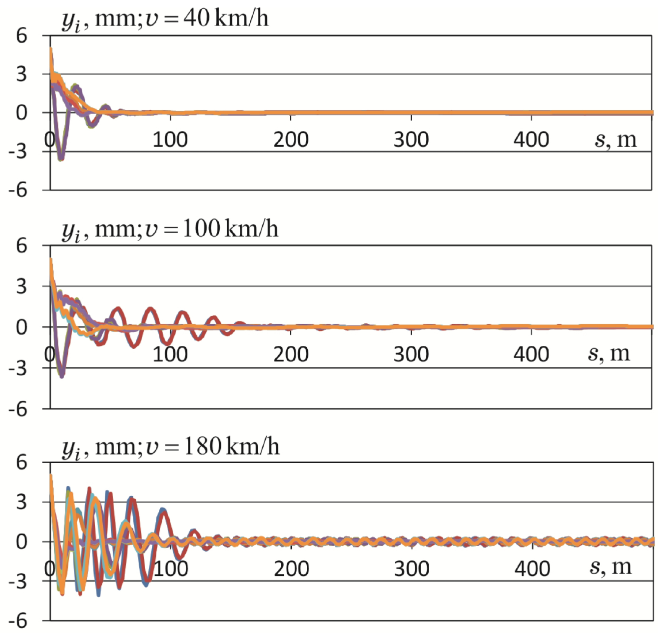


Fig. 2

At the train motion along the track with imperfections the main dynamic characteristics were calculated in accordance with the requirements of the Standard [10]. They are: car-body horizontal lateral $u_h$ and vertical $u_v$ accelerations; forces acting on the wheelsets in the lateral horizontal direction $H$ horizontal lateral and vertical dynamic indices $k_{dh}$ and $k_{dv}$; derailment stability coefficient $k_{st}$; horizontal and vertical ride quality coefficients $w_h, w_v$.

The results obtained have shown that at the train motion along the track maintained for the speedy motion all the dynamic characteristics insure train safe motion. It is proved by the data given on Fig. 3 – Fig. 5.

As the considered train is devoted for the passengers transportation, its ride quality is one of the main characteristics. At the carried simulations the processes of the middle car-body vertical and horizontal lateral accelerations have been used to determine the car ride quality factors [11]. The results obtained for the different speeds of train motion are shown on Fig. 6.

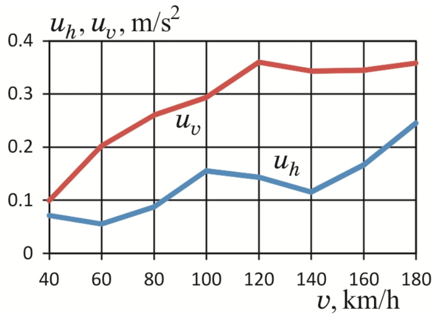


Fig. 3

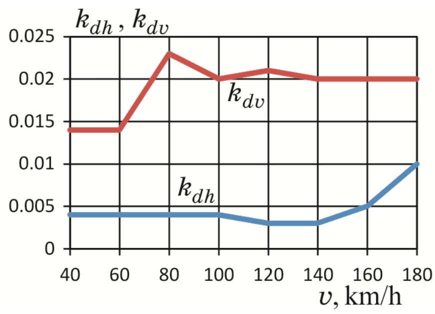


Fig. 4

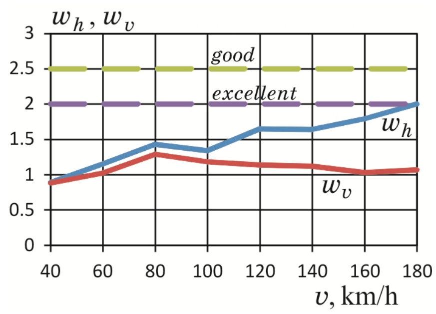


Fig. 5

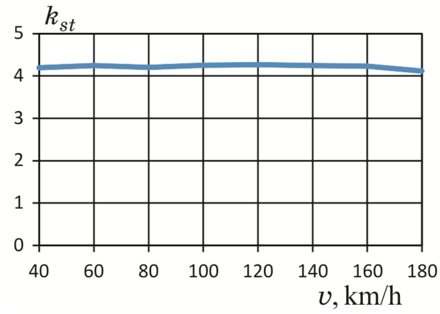


Fig. 6

From the given results it is seen that for the considered articulated train the ride quality values are in the range of the excellent quality which is insuring comfortable conditions for the travelling passengers.

**Conclusions.** Mathematical model describing the articulated train motion along the track of arbitrary shape in plan is developed. The model takes into account forces acting both in car elements and in intercar couplings. Some preliminary simulation results are given, which show the possibility to use the developed model and software for the estimation of the articulated train dynamic characteristics.

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