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TEMPERATURE INTEGRAL BRACKETS FOR A ONE-COMPONENT SYSTEM WITH SMALL INTERACTION

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Задача опису гідродинамічного етапу еволюції системи та відповідного розрахунку кінетичних коефіцієнтів системи є актуальною для статистичної фізики. Метод Чепмена–Енскога широко застосовується до відповідної задачі для різних систем, а поліноми Соніна широко використовуються для розрахунку наближених розв'язків для функції розподілу системи. Стандартна гідродинамічна теорія призводить до інтегральних рівнянь Фредгольма першого роду, для яких розв'язки, засновані на поліномах Соніна, вважаються збіжними. Слід зауважити, що часто аналітичні розрахунки обмежуються наближеннями одного або двох поліномів через громіздкість таких розрахунків та той факт, що збіжність розв'язків зі збільшенням кількості поліномів вважається досить швидкою.

Однак, числове дослідження відповідної збіжності представляє інтерес. Наприклад, числове дослідження відповідної збіжності для простого газу та газу твердих сфер для наближень навіть до 150 поліномів було проведено С. К. Лоялкою, Р. В. Томпсоном та Е. Л. Тіптоном на основі кінетичного рівняння Больцмана. Однак, нам невідомі роботи, де такі дослідження проводяться для систем, що описуються кінетичним рівнянням Ландау.

Як відомо, так звані системи з малою взаємодією описуються кінетичним рівнянням Ландау, яке містить інтеграл зіткнень Ландау. Наприклад, системи з кулонівською взаємодією описуються таким математичним апаратом. Зокрема, деякі попередні дослідження були присвячені повністю іонізованій двокомпонентній плазмі, і в більшості випадків використовувалися наближення одного або двох поліномів. У цій статті ми досліджуємо відповідні інтегральні дужки для однокомпонентної системи з малою взаємодією, але обчислено інтегральні дужки до наближення тринадцяти поліномів включно. В цій статті ми обмежуємося лише інтегральними дужками, необхідними для розрахунку температурної частини функції розподілу першого порядку по градієнтах. Отримано точні аналітичні результати для розглянутих інтегральних дужок. Отримані результати є важливими для подальшого числового дослідження збіжності результатів для теплопровідності системи зі збільшенням кількості поліномів у відповідних наближеннях. Дужки, необхідні для розрахунку швидкісної частини функції розподілу, можуть бути представлені в іншій статті.

Ключові слова: *інтегральні дужки, поліноми Соніна, інтеграл зіткнень Ландау, однокомпонентна система, мала взаємодія.*

The problem of description of the hydrodynamic stage of system evolution and the corresponding calculation of the system kinetic coefficients is urgent for statistical physics. The Chapman-Enskog method is widely applied to the corresponding problem for different systems, and the Sonine polynomials are widely used for the calculation of approximate solutions for the system distribution function. The standard hydrodynamic theory leads to Fredholm integral equations of the first kind, for which the solutions based on Sonine polynomials are considered to be convergent. It should be stressed that analytical calculations are often restricted to the one- or two-polynomial approximations because of the cumbersomeness of such calculations and the fact that the convergence of the solutions with increasing number of polynomials is considered to be rather fast.

However, the numerical investigation of the corresponding convergence is of interest. For example, a numerical investigation of the corresponding convergence for the simple and rigid-sphere gas approximations up to as many as 150 polynomials was made by S.K. Loyalka, R.V. Tompson, and E.L. Tipton on the basis of the Boltzmann kinetic equation. However, we do not know any works where such investigations would be made for systems described by the Landau kinetic equation.

As is known, the so-called systems with small interaction are described by the Landau kinetic equation, which contains the Landau collision integral. For example, systems with Coulomb interaction are described by this mathematical apparatus. In particular, some previous investigations were devoted to a completely ionized two-component plasma, and in most cases the one- or two-polynomial approximations were used. In this paper we investigate the corresponding integral brackets for a one-component system with small interaction, but the calculation of the integral brackets is made up to the thirteen-polynomial approximation. Here we restrict ourselves only to the integral brackets necessary for the calculation of the temperature part of the first-order-in-gradients distribution function. Exact analytical results for the integral brackets under consideration are obtained. The obtained results are important for further numerical investigation of the convergence of the results for the system thermal conductivity with increasing number of polynomials in corresponding approximations. The brackets that are necessary for the calculation of the velocity part of the distribution function may be the subject matter of another paper.

Keywords: *integral brackets, Sonine polynomials, Landau collision integral, one-component system, small interaction.*

Introduction. As is known [1], systems with small interaction are described by the Landau–Vlasov kinetic equation, which contains the Landau collision integral. For example, it is widely used in plasma investigations [2, 3]. The Chapman-Enskog method is widely used for the description of the hydrodynamic stage of system evolution, and the Sonine polynomial expansion is widely used to solve the corresponding integral equations for the system distribution function [2, 4]. The corresponding approximate analytical calculations are usually restricted to the one- or two-polynomial approximations [2], because the corresponding convergence rate is considered to be rather high, and the corresponding calculations are rather cumbersome. However, a calculation in mathematical packages may allow one to investigate the corresponding convergence numerically. For example, in [5] a numerical investigation of the convergence of Sonine polynomial expansions for a simple and a rigid-sphere gas is made up to the 150-polynomial approximation on the basis of the Boltzmann kinetic equation. However, we don't know any papers where a numerical investigation of Sonine polynomial expansions would be made for a system described by the Landau kinetic equation. So, this paper is devoted to the calculation of the integral brackets necessary for obtaining the temperature part of the first-order-in-gradients distribution function for a one-component system with weak interaction described by the Landau kinetic equation. In what follows, the integral brackets under consideration are called the temperature integral brackets.

Calculation of the temperature integral brackets. The Sonine polynomials are as follows [2]:

$$S_n^{\alpha}(x) = \frac{1}{n!} e^{x} x^{-\alpha} \frac{d^n}{dx^n}\left(e^{-x} x^{\alpha+n}\right) . \tag{1}$$

The temperature integral brackets are as follows [2]:

$$G_{js} = \left\{ p_l S_j^{3/2}\left(\beta\varepsilon_p\right), p_l S_s^{3/2}\left(\beta\varepsilon_p\right)\right\} \tag{2}$$

where

$$\begin{gathered} \{g,h\} = C\int d^3p d^3p' e^{-\beta\varepsilon_p - \beta\varepsilon_{p'}} D_{nk,pp'} \left\{\frac{\partial g(\vec{p})}{\partial p_n} - \frac{\partial g(\vec{p}')}{\partial p'_n}\right\}\left\{\frac{\partial h(\vec{p})}{\partial p_k} - \frac{\partial h(\vec{p}')}{\partial p'_k}\right\}, \\ \beta = \frac{1}{T}, \; \varepsilon_p = \frac{p^2}{2m}, \; D_{nk,pp'} = \frac{\left|\vec{p}-\vec{p}'\right|^2 \delta_{nk} - (p-p')_n (p-p')_k}{\left|\vec{p}-\vec{p}'\right|^3}, \end{gathered} \tag{3}$$

p is the momentum, T is the system temperature, m is the mass of one particle, and C is a constant. The results of the previous investigation [2] are based on the calculation of G_{js} up to the two-polynomial approximation. In this paper we calculate the corresponding brackets up to the thirteen-polynomial approximation.

First of all, we introduce dimensionless variables, $\vec{q}$ and $\vec{q}'$:

$$\vec{p} = \sqrt{mT}\vec{q}, \; \vec{p}' = \sqrt{mT}\vec{q}' . \tag{3}$$

On substitution, we obtain

$$G_{js} = C\left(mT\right)^{5/2} \int d^3q d^3q' \exp\left(-\left(q^2+q'^2\right)/2\right) D_{nk,qq'} B^{(j)}_{nl,qq'} B^{(s)}_{kl,qq'} \tag{4}$$

where

$$B^{(j)}_{nl,qq'} = \frac{\partial q_l S_j^{3/2}\left(q^2/2\right)}{\partial q_n} - \frac{\partial q'_l S_j^{3/2}\left(q'^2/2\right)}{\partial q'_n} =$$
$$= \delta_{nl}\left(S_j^{3/2}\left(\frac{q^2}{2}\right) - S_j^{3/2}\left(\frac{q'^2}{2}\right)\right) + q_l q_n \frac{\partial S_j^{3/2}\left(q^2/2\right)}{\partial\left(q^2/2\right)} - q'_l q'_n \frac{\partial S_j^{3/2}\left(q'^2/2\right)}{\partial\left(q'^2/2\right)}. \tag{5}$$

By a straightforward calculation on the basis of (5) it may be shown that

$$B^{(j)}_{nl,qq'} B^{(s)}_{kl,qq'} = \delta_{nk}\Delta_{jqq'}\Delta_{sqq'} + q_n q_k\left(\Delta_j\Theta_{sq} + \Theta_{jq}\Delta_s + q^2\Theta_{jq}\Theta_{sq}\right) -$$
$$-\left(q,q'\right)\left[q_n q'_k \Theta_{jq}\Theta_{sq'} + q_k q'_n \Theta_{jq'}\Theta_{sq}\right] \tag{6}$$

where

$$\Delta_{jqq'} = S_j^{3/2}\left(\frac{q^2}{2}\right) - S_j^{3/2}\left(\frac{q'^2}{2}\right),\ \Theta_{jq} = \frac{\partial S_j^{3/2}\left(q^2/2\right)}{\partial\left(q^2/2\right)}. \tag{7}$$

We introduce spherical coordinates:

$$\vec{x} = \vec{q} + \vec{q}',\ \vec{y} = \vec{q} - \vec{q}',\ \vec{x} = \left(x\sin\theta_x\cos\varphi_x, x\sin\theta_x\sin\varphi_x, x\cos\theta_x\right),$$
$$\vec{y} = \left(y\sin\theta_y\cos\varphi_y, y\sin\theta_y\sin\varphi_y, y\cos\theta_y\right). \tag{8}$$

On the basis of (8) and (3) a straightforward calculation leads to

$$q^2 = \frac{x^2 + 2xy\Omega + y^2}{4},\ q'^2 = \frac{x^2 - 2xy\Omega + y^2}{4},$$
$$\left(q,q'\right) = \frac{x^2 - y^2}{4},\ D_{nk,qq'}\delta_{nk} = \frac{2}{y}, \tag{9}$$
$$D_{nk,qq'} q_n q_k = D_{nk,qq'} q'_n q'_k = D_{nk,qq'} q_n q'_k = D_{nk,qq'} q'_n q_k = \frac{x^2\left(1-\Omega^2\right)}{4y},$$

where

$$\Omega = \sin\theta_x\cos\varphi_x\sin\theta_y\cos\varphi_y + \sin\theta_x\sin\varphi_x\sin\theta_y\sin\varphi_y + \cos\theta_x\cos\theta_y. \tag{10}$$

On the basis of (6), (7), and (9), a straightforward calculation yields

$$D_{nk,qq'} B^{(j)}_{nl,qq'} B^{(s)}_{kl,qq'} = \frac{2}{y}\Delta_{jqq'}\Delta_{sqq'} +$$
$$+\frac{x^2 - x^2\Omega^2}{4y}\left[\Delta_{jqq'}\left(\Theta_{sq} - \Theta_{sq'}\right) + \Delta_{sqq'}\left(\Theta_{jq} - \Theta_{jq'}\right)\right] + \tag{11}$$
$$+\frac{x^2 - x^2\Omega^2}{4y}\left[q^2\Theta_{jq}\Theta_{sq} + q'^2\Theta_{jq'}\Theta_{sq'} - \left(q,q'\right)\left\{\Theta_{jq}\Theta_{sq'} + \Theta_{jq'}\Theta_{sq}\right\}\right].$$

where according to (9)

$$\Delta_{jqq'} = S_j^{3/2}\left(\frac{x^2+2xy\Omega+y^2}{8}\right) - S_j^{3/2}\left(\frac{x^2-2xy\Omega+y^2}{8}\right),$$

$$\Theta_{jq} = \left.\frac{\partial S_j^{3/2}\left(q^2/2\right)}{\partial\left(q^2/2\right)}\right|_{q^2=\frac{x^2+2xy\Omega+y^2}{4}},\ \Theta_{jq'} = \left.\frac{\partial S_j^{3/2}\left(q'^2/2\right)}{\partial\left(q'^2/2\right)}\right|_{q'^2=\frac{x^2-2xy\Omega+y^2}{4}}. \qquad (12)$$

According to (4) and (8), the temperature integral brackets take the form

$$G_{js} = \frac{C\left(mT\right)^{5/2}}{8}\int_0^\infty dx e^{-x^2/4}\int_0^\infty dx e^{-y^2/4}\int d\Omega x^2 y^2 D_{nk,qq'} B^{(j)}_{nl,qq'} B^{(s)}_{kl,qq'}\,. \qquad (13)$$

where the product $D_{nk,qq'} B^{(j)}_{nl,qq'} B^{(s)}_{kl,qq'}$ is given by (11) and (12) with account for (10), and

$$\int d\Omega = \int_0^\pi d\theta_x \sin\theta_x \int_0^\pi d\theta_y \sin\theta_y \int_0^{2\pi} d\varphi_x \int_0^{2\pi} d\varphi_y\,. \qquad (14)$$

The product $x^2 y^2 D_{nk,qq'} B^{(j)}_{nl,qq'} B^{(s)}_{kl,qq'}$ is a sum of some terms that have the form $ax^b y^c \Omega^d$ where a is a constant, b is an even integer not less than 2, c is an odd integer not less than 1, and d is an even integer not less than 0.

In Appendix A on the basis of the results described in [6, 7] it is shown that

$$\int d\Omega\Omega^{2n} = \int_0^\pi d\theta_x \sin\theta_x \int_0^\pi d\theta_y \sin\theta_y \int_0^{2\pi} d\varphi_x \int_0^{2\pi} d\varphi_y \times$$

$$\times\left(\sin\theta_x\cos\varphi_x\sin\theta_y\cos\varphi_y + \sin\theta_x\sin\varphi_x\sin\theta_y\sin\varphi_y + \cos\theta_x\cos\theta_y\right)^{2n} = \qquad (15)$$

$$= \frac{16\pi^2}{2n+1} = 16\pi^2\int_0^1 d\Omega\Omega^{2n}$$

where Ω is considered as a one-dimensional variable. So,

$$G_{js} = 2\pi^2 C\left(mT\right)^{5/2}\int_0^\infty dx x^2 e^{-x^2/4}\int_0^\infty dx y^2 e^{-y^2/4}\int_0^1 d\Omega D_{nk,qq'} B^{(j)}_{nl,qq'} B^{(s)}_{kl,qq'}\,, \qquad (16)$$

$D_{nk,qq'} B^{(j)}_{nl,qq'} B^{(s)}_{kl,qq'}$ is given in (11), (12) with account for (10).

A further calculation of (16) in the general case is rather cumbersome, and it is made in the Wolfram Mathematica package. The explicit analytical results for G_{js} are given in Appendix B. The Wolfram Mathematica code is given in Appendix C.

Conclusions. The exact analytical results for the temperature integral brackets are calculated up to the thirteen-polynomial approximation. The results up to the two-polynomial approximation coincide with the previous investigation [2], and the other results are new ones. The obtained integral brackets are important for further numerical investigation of the convergence of the results for the thermal conductivity of the systems under consideration with increasing number of polynomials.

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Appendix A. Proof of Eq. (15). The integral (15) may be rewritten as

$$f(n)=\int d\Omega\Omega^{2n}=\int_0^\pi d\theta_x \sin\theta_x \int_0^\pi d\theta_y \sin\theta_y \int_0^{2\pi} d\varphi_x \int_0^{2\pi} d\varphi_y \times$$
$$\times\left(\sin\theta_x \sin\theta_y \cos\left(\varphi_x-\varphi_y\right)+\cos\theta_x \cos\theta_y\right)^{2n}=$$
$$=\sum_{k=0}^{2n} C_{2n}^k \int_0^\pi d\theta_x \sin^{k+1}\theta_x \cos^{2n-k}\theta_x \int_0^\pi d\theta_y \sin^{k+1}\theta_y \cos^{2n-k}\theta_y \times \tag{A.1}$$
$$\times\int_0^{2\pi} d\varphi_x \int_0^{2\pi} d\varphi_y \cos^k\left(\varphi_x-\varphi_y\right)$$

where $C_{2n}^k=(2n)!/(k!\cdot(2n-k)!)$. On the basis of the tabulated integral [6]

$$\int_0^{\pi/2} dx \sin^{\mu-1}x \cos^{\nu-1}x=\frac{1}{2}B\left(\frac{\mu}{2},\frac{\nu}{2}\right)=\frac{1}{2}\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\nu}{2}\right)\Big/\Gamma\left(\frac{\mu+\nu}{2}\right) \tag{A.2}$$

we obtain

$$\int_0^\pi d\theta_x \sin^{k+1}\theta_x \cos^{2n-k}\theta_x=\int_0^{\pi/2} d\theta_x \sin^{k+1}\theta_x \cos^{2n-k}\theta_x+$$
$$+\int_{\pi/2}^{\pi} d\theta_x \sin^{k+1}\theta_x \cos^{2n-k}\theta_x=\left\{\alpha_x=\theta_x-\frac{\pi}{2}\right\}=$$
$$=\int_0^{\pi/2} d\theta_x \sin^{k+1}\theta_x \cos^{2n-k}\theta_x+(-1)^{2n-k}\int_0^{\pi/2} d\alpha_x \cos^{k+1}\alpha_x \sin^{2n-k}\alpha_x= \tag{A.3}$$
$$=\frac{1+(-1)^{2n-k}}{2}\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{2n-k+1}{2}\right)\Big/\Gamma\left(n+\frac{3}{2}\right).$$

First of all, it is obvious that this integral is non-zero only if k is an even number. Let us take

$$k=2m\ ,\ m=\overline{0,n}\,. \tag{A.4}$$

Then on the basis of obvious expressions

$$\Gamma\left(n+\frac{3}{2}\right)=\frac{(2n+1)!!\cdot\sqrt{\pi}}{2^{n+1}}\ ,\ \Gamma\left(n+\frac{1}{2}\right)=\frac{(2n-1)!!\cdot\sqrt{\pi}}{2^{n}}\ ,\ (-1)!!=0!!=1 \tag{A.5}$$

one can conclude that

$$\int_0^{\pi} d\theta_x \sin^{k+1}\theta_x \cos^{2n-k}\theta_x = \begin{cases} 2^{m+1} m! \dfrac{(2n-2m-1)!!}{(2n+1)!!}, k = 2m, m = \overline{0,n} \\ 0, k = 2m+1, m = \overline{0,n-1} \end{cases}. \quad \text{(A.6)}$$

As for the integrals over the variables φ_x and φ_y, we have

$$\begin{aligned} &\int_0^{2\pi} d\varphi_x \int_0^{2\pi} d\varphi_y \cos^k\left(\varphi_x - \varphi_y\right) = \\ &= \sum_{l=0}^{k} C_l^k \int_0^{2\pi} \cos^l \varphi_x \sin^{k-l} \varphi_x d\varphi_x \int_0^{2\pi} d\varphi_y \sin^{k-l} \varphi_y \cos^l \varphi_y , \end{aligned} \quad \text{(A.7)}$$

and according to (A.6) and (A.1), it makes sense to take the integral (A.7) only for even $k = 2m,\ m = \overline{0,n}$. According to (A.2),

$$\begin{aligned} &\int_0^{2\pi} \cos^l \varphi_x \sin^{2m-l} \varphi_x d\varphi_x = \int_0^{\pi/2} \cos^l \varphi_x \sin^{2m-l} \varphi_x d\varphi_x + \\ &+ \int_{\pi/2}^{\pi} \cos^l \varphi_x \sin^{2m-l} \varphi_x d\varphi_x + \int_{\pi}^{3\pi/2} \cos^l \varphi_x \sin^{2m-l} \varphi_x d\varphi_x + \\ &+ \int_{3\pi/2}^{2\pi} \cos^l \varphi_x \sin^{2m-l} \varphi_x d\varphi_x = \int_0^{\pi/2} \cos^l \varphi_x \sin^{2m-l} \varphi_x d\varphi_x + \\ &+ (-1)^l \int_0^{\pi/2} \sin^l \varphi_x \cos^{2m-l} \varphi_x d\varphi_x + (-1)^l (-1)^{2m-l} \int_0^{\pi/2} \cos^l \varphi_x \sin^{2m-l} \varphi_x d\varphi_x + \\ &+ (-1)^{2m-l} \int_0^{\pi/2} \sin^l \varphi_x \cos^{2m-l} \varphi_x d\varphi_x = \\ &= \frac{1}{2m!}\left(2 + (-1)^l + (-1)^{2m-l}\right)\Gamma\left(\frac{l+1}{2}\right)\Gamma\left(\frac{2m-l+1}{2}\right), \end{aligned} \quad \text{(A.8)}$$

here, the identity $\Gamma(m+1) = m!$ is used. As can be seen, the integral (A.8) is nonzero only if l is an even number: $l = 2z, z = \overline{0,m}$. On the basis of (A.5) we see that

$$\int_0^{2\pi} \cos^l \varphi_x \sin^{2m-l} \varphi_x d\varphi_x = \begin{cases} 0, l = 2z+1, z = \overline{0,m-1} \\ \dfrac{\pi(2m-2z-1)!!(2z-1)!!}{2^{m-1} m!}, l = 2z, z = \overline{0,m} \end{cases}. \quad \text{(A.9)}$$

On the basis of (A.9) and (A.6) one can conclude that the integral (A.1) is as follows:

$$f(n) = \pi^2 \sum_{m=0}^{n} C_{2n}^{2m} \left(2^{m+1} m! \frac{(2n-2m-1)!!}{(2n+1)!!}\right)^2 \frac{1}{\left(2^{m-1} m!\right)^2} \times \quad \text{(A.10)}$$

$$\times\sum_{z=0}^{m} C_{2z}^{2m}\left((2m-2z-1)!!(2z-1)!!\right)^2 =$$

$$=16\pi^2\sum_{m=0}^{n}\sum_{z=0}^{m}\frac{(2n)!}{(2n-2m)!(2m-2z)!(2z)!}\times$$

$$\times\left(\frac{(2n-2m-1)!!(2m-2z-1)!!(2z-1)!!}{(2n+1)!!}\right)^2 .$$

On the basis of the obvious identity

$$(2n)! = (2n)!!(2n-1)!! . \tag{A.11}$$

we rewrite $f(n)$ as

$$f(n)=\frac{16\pi^2}{2n+1}\frac{(2n)!!}{(2n+1)!!}\sum_{m=0}^{n}\frac{(2n-2m-1)!!}{(2n-2m)!!}\sum_{z=0}^{m}\frac{(2m-2z-1)!!(2z-1)!!}{(2m-2z)!!(2z)!!} . \tag{A.12}$$

First of all, let us prove that

$$\sum_{z=0}^{m}\frac{(2m-2z-1)!!(2z-1)!!}{(2m-2z)!!(2z)!!}=1 . \tag{A.13}$$

On the basis of the well-known expressions

$$(2n-1)!!=\frac{(2n)!}{2^n n!} ,\ (2n)!!=2^n n! \tag{A.14}$$

we rewrite (A.13) as

$$\sum_{z=0}^{m}\frac{(2m-2z-1)!!(2z-1)!!}{(2m-2z)!!(2z)!!}=\frac{1}{2^{2m}}\sum_{z=0}^{m}\frac{(2m-2z)!}{(m-z)!(m-z)!}\frac{(2z)!}{z!z!}=$$
$$=\frac{1}{2^{2m}}\sum_{z=0}^{m}C_{2m-2z}^{m-z}C_{2z}^{z} . \tag{A.15}$$

As is known [7],

$$\sum_{z=0}^{m}C_{2m-2z}^{m-z}C_{2z}^{z}=2^{2m} , \tag{A.16}$$

So, the expression (A.13) is proved. Let us prove that

$$\sum_{m=0}^{n}\frac{(2n-2m-1)!!}{(2n-2m)!!}=\{z=n-m\}=\sum_{z=0}^{n}\frac{(2z-1)!!}{(2z)!!}=\frac{(2n+1)!!}{(2n)!!} . \tag{A.17}$$

Let us use the method of mathematical induction. The base case is the obvious fact that (A.17) holds for $n=0$. The inductive hypothesis is (A.17). The inductive step consists in the proof of the identity

$$\sum_{m=0}^{n+1}\frac{(2z-1)!!}{(2z)!!}=\frac{(2n+3)!!}{(2n+2)!!} \tag{A.18}$$

on the basis of (A.17):

$$\sum_{m=0}^{n+1}\frac{(2z-1)!!}{(2z)!!}=\sum_{z=0}^{n}\frac{(2z-1)!!}{(2z)!!}+\frac{(2n+1)!!}{(2n+2)!!}=\frac{(2n+1)!!}{(2n)!!}+\frac{(2n+1)!!}{(2n+2)!!}= \\ =\frac{(2n+1)!!}{(2n)!!}\left(1+\frac{1}{2n+2}\right)=\frac{(2n+1)!!}{(2n)!!}\left(\frac{2n+3}{2n+2}\right)=\frac{(2n+3)!!}{(2n+2)!!}, \tag{A.19}$$

which was to be proved. So, the identity (A.17) is proved. On the basis of (A.13) and (A.17) one can conclude that (A.12) may be rewritten as

$$f(n)=\frac{16\pi^2}{2n+1}\frac{(2n)!!}{(2n+1)!!}\frac{(2n+1)!!}{(2n)!!}=\frac{16\pi^2}{2n+1}, \tag{A.20}$$

which proves the statement, see (A.1) and (15).

Appendix B. Explicit results for G_{js}. Let us denote

$$G_{js}=C(\pi mT)^{5/2}\,Y_{js}/8. \tag{B.1}$$

Due to the symmetry of the integral brackets, we have

$$Y_{js}=Y_{sj}. \tag{B.2}$$

The following explicit expressions for Y_{js} are obtained:

$$\begin{gathered}
Y_{j0}=0,\, Y_{11}=512,\; Y_{21}=384,\; Y_{22}=1440,\; Y_{31}=240,\; Y_{32}=1236, \\
Y_{33}=\frac{5657}{2},\; Y_{41}=140,\; Y_{42}=885,\; Y_{43}=\frac{20349}{8},\; Y_{44}=\frac{149749}{32},\; Y_{51}=\frac{315}{4}, \\
Y_{52}=\frac{9345}{16},\; Y_{53}=\frac{249285}{128},\; Y_{54}=\frac{2204625}{512},\; Y_{55}=\frac{57292281}{8192},\; Y_{61}=\frac{693}{16}, \\
Y_{62}=\frac{23499}{64},\; Y_{63}=\frac{705635}{512},\; Y_{64}=\frac{7034355}{2048},\; Y_{65}=\frac{213752391}{32768}, \\
Y_{66}=\frac{1280638685}{131072},\; Y_{71}=\frac{3003}{128},\; Y_{72}=\frac{114345}{512},\; Y_{73}=\frac{3800349}{4096}, \\
Y_{74}=\frac{41702465}{16384},\; Y_{75}=\frac{1402916745}{262144},\; Y_{76}=\frac{9642765231}{1048576}, \\
Y_{77}=\frac{109135301905}{8388608},\; Y_{81}=\frac{6435}{512},\; Y_{82}=\frac{271557}{2048},\; Y_{83}=\frac{9876141}{16384}, \\
Y_{84}=\frac{117773397}{65536},\; Y_{85}=\frac{4298885745}{1048576},\; Y_{86}=\frac{32305668675}{4194304}, \\
Y_{87}=\frac{413557727313}{33554432},\; Y_{88}=\frac{2243029645077}{134217728},\; Y_{91}=\frac{109395}{16384}, \\
Y_{92}=\frac{5064345}{65536},\; Y_{93}=\frac{199832061}{524288},\; Y_{94}=\frac{2567855961}{2097152}, \\
Y_{95}=\frac{100626992865}{33554432},\; Y_{96}=\frac{812212203215}{134217728},\; Y_{97}=\frac{11262985040385}{1073741824}, \\
Y_{98}=\frac{68334759105177}{4294967296},\; Y_{99}=\frac{2869197779473001}{137438953472},\; Y_{10,1}=\frac{230945}{65536},
\end{gathered} \tag{B.3}$$

$$Y_{10,2}=\frac{11632335}{262144},\ Y_{10,3}=\frac{494634855}{2097152},\ Y_{10,4}=\frac{6806334535}{8388608},$$

$$Y_{10,5}=\frac{284401395531}{134217728},\ Y_{10,6}=\frac{2442996753561}{536870912},\ Y_{10,7}=\frac{36111796183715}{4294967296},$$

$$Y_{10,8}=\frac{235634511362295}{17179869184},\ Y_{10,9}=\frac{10969156676888595}{549755813888},$$

$$Y_{10,10}=\frac{56081014779421189}{2199023255552},\ Y_{11,1}=\frac{969969}{524288},\ Y_{11,3}=\frac{2405778375}{16777216},$$

$$Y_{11,4}=\frac{35273210115}{67108864},\ Y_{11,5}=\frac{1563806425467}{1073741824},\ Y_{11,6}=\frac{14213345746941}{4294967296},$$

$$Y_{11,7}=\frac{222119161169283}{34359738368},\ Y_{11,8}=\frac{1535881558864275}{137438953472},$$

$$Y_{11,9}=\frac{76451461175367555}{4398046511104},\ Y_{11,10}=\frac{430170690626461113}{17592186044416},$$

$$Y_{11,11}=\frac{4305352430573002713}{140737488355328},\ Y_{12,1}=\frac{2028117}{2097152},\ Y_{12,2}=\frac{118600755}{8388608},$$

$$Y_{12,3}=\frac{5764508971}{67108864},\ Y_{12,4}=\frac{89684596755}{268435456},\ Y_{12,5}=\frac{4202216808855}{4294967296},$$

$$Y_{12,6}=\frac{40245601722469}{17179869184},\ Y_{12,7}=\frac{661544369311815}{137438953472},$$

$$Y_{12,8}=\frac{4811057022438387}{549755813888},\ Y_{12,9}=\frac{252577679154968335}{17592186044416},$$

$$Y_{12,10}=\frac{1512441361404919665}{70368744177664},\ Y_{12,11}=\frac{16556070452511642837}{562949953421312},$$

$$Y_{12,12}=\frac{81384782699234451365}{2251799813685248},\ Y_{13,1}=\frac{16900975}{33554432},\ Y_{13,2}=\frac{1056648957}{134217728},$$

$$Y_{13,3}=\frac{54556259121}{1073741824},\ Y_{13,4}=\frac{897497127917}{4294967296},\ Y_{13,5}=\frac{44300679154485}{68719476736},$$

$$Y_{13,6}=\frac{445637163040875}{274877906944},\ Y_{13,7}=\frac{7677511852434853}{2199023255552},$$

$$Y_{13,8}=\frac{58454110673432397}{8796093022208},\ Y_{13,9}=\frac{3214110272424444477}{281474976710656},$$

$$Y_{13,10}=\frac{20219941595148385055}{1125899906842624},\ Y_{13,11}=\frac{234612748994374719375}{9007199254740992},$$

$$Y_{13,12}=\frac{125468031095156808867}{36028797018963968},\ Y_{13,13}=\frac{24300600849403343223937}{576460752303423488}.$$

Appendix C. The listing of the Wolfram Mathematica code

```
eN = 13;
Sonine[n_, \[Alpha]_,
  z_] = ((1/n!)*Exp[z]*D[Exp[-z]*z^(\[Alpha] + n), {z, n}])/
  z^\[Alpha];
DerList[n_] :=
 Module[{en = n}, Thes[u_] = Sonine[en, 3/2, u];
  TheSS[u_] = Simplify[Thes[u]]; zy = CoefficientList[TheSS[u], u];
```

```
  zy];
S[n_, z_] :=
  Module[{en = n, ez = z}, ksy = DerList[en];
   ThePolynom = Sum[ksy[[j + 1]]*ez^j, {j, 0, en}]; ThePolynom];
DDerList[n_] :=
  Module[{en = n}, ws[u_] = D[S[n, u], u]; wws[u_] = Simplify[ws[u]];
   zy = CoefficientList[wws[u], u]; zy];
DS[n_, z_] :=
  Module[{en = n, ez = z}, ksy = DDerList[en];
   ThePolynom = Sum[ksy[[j + 1]]*ez^j, {j, 0, en - 1}]; ThePolynom];
Q[x_, y_, \[CapitalOmega]_] = (x^2 + 2*x*y*\[CapitalOmega] + y^2)/8;
R[x_, y_, \[CapitalOmega]_] = (x^2 - 2*x*y*\[CapitalOmega] + y^2)/8;
G = IdentityMatrix[eN + 1];
For[ii = 1, ii <= eN + 1, ii++,
  For[jj = 1, jj <= ii, jj++, j = ii - 1; s = jj - 1;
    UnderIntergand[x_, y_, \[CapitalOmega]_] =
     Expand[(2/y)*(S[j, Q[x, y, \[CapitalOmega]]] -
          S[j, R[x, y, \[CapitalOmega]]])*(S[s,
           Q[x, y, \[CapitalOmega]]] -
          S[s, R[x, y, \[CapitalOmega]]]) +
            ((x^2 - x^2*\[CapitalOmega]^2)/(4*
            y))*((S[j, Q[x, y, \[CapitalOmega]]] -
             S[j, R[x, y, \[CapitalOmega]]])*(DS[s,
              Q[x, y, \[CapitalOmega]]] -
             DS[s, R[x, y, \[CapitalOmega]]]) + (DS[j,
              Q[x, y, \[CapitalOmega]]] -
             DS[j, R[x, y, \[CapitalOmega]]])*(S[s,
              Q[x, y, \[CapitalOmega]]] -
             S[s, R[x, y, \[CapitalOmega]]]) +
          2*Q[x, y, \[CapitalOmega]]*DS[j, Q[x, y, \[CapitalOmega]]]*
           DS[s, Q[x, y, \[CapitalOmega]]] +

          2*R[x, y, \[CapitalOmega]]*DS[j, R[x, y, \[CapitalOmega]]]*
           DS[s, R[x, y, \[CapitalOmega]]] - ((x^2 - y^2)/
             4)*(DS[j, Q[x, y, \[CapitalOmega]]]*
              DS[s, R[x, y, \[CapitalOmega]]] +
             DS[s, Q[x, y, \[CapitalOmega]]]*
              DS[j, R[x, y, \[CapitalOmega]]]))];
    G[[ii, jj]] =
     16*Pi^2*Integrate[
       x^2*Exp[-(x^2/4)]*y^2*Exp[-(y^2/4)]*
        UnderIntergand[x, y, \[CapitalOmega]], {x, 0, Infinity}, {y,
        0, Infinity}, {\[CapitalOmega], 0, 1}];
    G[[jj, ii]] = G[[ii, jj]];
    Print["G[", ii - 1, ",", jj - 1, "]=", G[[ii, jj]]]; ]; ];
```

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