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VELOCITY INTEGRAL BRACKETS FOR A ONE-COMPONENT SYSTEM WITH SMALL INTERACTION

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Стандартним та широко використовуваним в літературі підходом до опису гідродинамічного етапу еволюції системи є метод Чепмена–Енскога, в рамках якого функція розподілу системи обчислюється в теорії збурень за малими градієнтами. Кінетичні коефіцієнти системи обчислюються на основі отриманої функції розподілу першого порядку по градієнтах, температурна та швидкісна частини якої є розв'язками інтегральних рівнянь Фредгольма першого роду. В літературі вважається, що стандартним підходом до розв'язання відповідних інтегральних рівнянь є наближений пошук їх розв'язку за допомогою методу Галеркіна, що базується на штучно обірваному розвиненні за поліномами Соніна.

Так звані інтегральні дужки необхідні для обчислення коефіцієнтів при поліномах та обчислення інтегральних дужок є найбільш громіздким етапом обчислення кінетичних коефіцієнтів. У літературі вважається, що розв'язок рівняння Фредгольма першого роду швидко збігається зі збільшенням кількості поліномів, тому при отриманні відповідних аналітичних розв'язків дослідники часто обмежуються наближеннями одного або двох поліномів. Однак, для низки систем числове дослідження відповідної збіжності проводиться на основі відповідних числових розрахунків для наближень великої кількості поліномів. Нам невідомі роботи, де відповідне числове дослідження наведено для системи з малою взаємодією, що описується кінетичним рівнянням Ландау, яке містить інтеграл зіткнень Ландау.

У нашій попередній статті ми обчислили так звані температурні інтегральні дужки для однокомпонентної системи з малою взаємодією до наближення тринадцяти поліномів включно, відповідні інтегральні дужки необхідні для розрахунку теплопровідності системи. У цій статті нами обчислено так звані швидкісні інтегральні дужки для однокомпонентної системи з малою взаємодією до наближення тринадцяти поліномів включно, відповідні інтегральні дужки необхідні для розрахунку в'язкості системи. Отримані результати є важливими для подальшого числового дослідження збіжності розв'язків для теплопровідності та в'язкості системи зі збільшенням кількості поліномів. Відповідне числове дослідження з детальним розрахунком функції розподілу першого порядку малості за градієнтами може бути наведено в іншій статті.

Ключові слова: *інтегральні дужки, поліноми Соніна, інтеграл зіткнень Ландау, однокомпонентна система, мала взаємодія.*

The standard and widely used approach to the description of the hydrodynamic stage of the system evolution is the Chapman–Enskog method, in the framework of which the system distribution function is calculated in a perturbation theory in small gradients. The system kinetic coefficients are calculated on the basis of the obtained first-order-in-gradients distribution function, the temperature and velocity parts of which are the solutions of Fredholm integral equations of the first kind. In the literature it is considered that the standard approach to the solution of the corresponding integral equations is an approximate search for their solution with the help of the Galerkin method based on an artificially truncated Sonine polynomial expansion.

The so-called integral brackets are needed in order to calculate the coefficients multiplying the polynomials, and the calculation of the integral brackets is the most cumbersome stage of the kinetic coefficient calculation. In the literature it is assumed that the solution of a Fredholm equation of the first kind converges fast with increasing number of polynomials, that is why the corresponding analytical solutions are often restricted to the one- or two-polynomial approximations. However, for a number of systems the numerical investigation of the corresponding convergence is made on the basis of the corresponding numerical calculations for many-polynomial approximations. We do not know any works where the corresponding numerical investigation would be provided for a system with small interaction described by the Landau kinetic equation which contains the Landau collision integral.

In our previous paper, we calculated the so-called temperature integral brackets for a one-component system with small interaction up to the thirteen-polynomial approximation; the corresponding integral brackets are needed in order to calculate the system thermal conductivity. In this paper, we calculate the so-called velocity integral brackets for a one-component system with small interaction up to the thirteen-polynomial approximation, the corresponding integral brackets are needed in order to calculate the system viscosity. The obtained results are important for a further numerical investigation of the convergence of the solutions for the system thermal conductivity and viscosity with increasing number of polynomials. The corresponding numerical investigation with a

detailed calculation of the first-order-in-gradients distribution function may be given in another paper.

Keywords: *integral brackets, Sonine polynomials, Landau collision integral, one-component system, small interaction.*

Introduction. We consider a one-component system with small interaction. As is known, such a system is described by the Landau-Vlasov kinetic equation which contains the Landau collision integral [1]. As is known, the so-called Vlasov term which contains the self-consistent field has no effect on the system kinetic coefficients (see, for example, [2]), so the system kinetic coefficients (viscosity, thermal conductivity) may be calculated on the basis of the Landau kinetic equation without taking into account the Vlasov self-consistent term.

The system kinetic coefficients may be obtained on the basis of the Chapman-Enskog method (see, for example, [3]), on the basis of which the system distribution function is sought in the perturbation theory in small gradients, and the system kinetic coefficients may be obtained on the basis of the first-order-in-gradients distribution function, see, for example, [4]. The first-order-in-gradients distribution function is the solution of the Fredholm integral equation of the first kind. The Sonine polynomial expansion is widely used for obtaining of the approximate analytical solution of the corresponding equation, see, for example, [4]. Usually analytical calculations devoted to the Fredholm equations of the first and even of the second kind are restricted to one- or two-polynomial approximations, see [4 – 7]. Moreover, it is considered that in the case of the Fredholm equation of the first kind the solution converges rapidly with increasing of the number of polynomials. However, the numerical investigation of the corresponding convergence is of interest. For example, the corresponding investigation is made for binary gas mixtures of rigid-sphere molecules [8] and for the solution of the chemical kinetic Boltzmann equation [9].

The most cumbersome part of the calculation of the first-order-in-gradients distribution function is the calculation of the so-called integral brackets. In our recent paper [10] we calculated the temperature integral brackets which are necessary for the obtaining of the system thermal conductivity. This paper is devoted to the calculation of the velocity integral brackets which are necessary for the obtaining of the system viscosity. The investigation is made up to the thirteen-polynomial approximation.

Calculation of the velocity integral brackets. The Sonine polynomials are as follows, see, for example, [10]:

$$S_n^{\alpha}(x)=\frac{1}{n!}e^{x}x^{-\alpha}\frac{d^n}{dx^n}\left(e^{-x}x^{\alpha+n}\right). \tag{1}$$

The temperature integral brackets are as follows [4]

$$H_{js}=\left\{\left(p_l p_m-\frac{1}{3}\delta_{lm}p^2\right)S_j^{5/2}\left(\beta\varepsilon_p\right),\left(p_l p_m-\frac{1}{3}\delta_{lm}p^2\right)S_s^{5/2}\left(\beta\varepsilon_p\right)\right\} \tag{2}$$

where δ_{lm} is the Kronecker delta and

$$\{g,h\}=C\int d^3p\,d^3p'e^{-\beta\varepsilon_p-\beta\varepsilon_{p'}}D_{nk,pp'}\left\{\frac{\partial g(\vec{p})}{\partial p_n}-\frac{\partial g(\vec{p}')}{\partial p_n'}\right\}\left\{\frac{\partial h(\vec{p})}{\partial p_k}-\frac{\partial h(\vec{p}')}{\partial p_k'}\right\}, \tag{3}$$

$$\beta = \frac{1}{T}\,,\ \varepsilon_p = \frac{p^2}{2m}\,,\ D_{nk,pp'} = \frac{\left|\vec{p}-\vec{p}'\right|^2 \delta_{nk} - \left(p-p'\right)_n \left(p-p'\right)_k}{\left|\vec{p}-\vec{p}'\right|^3}\,,$$

here T is the system temperature, m is the mass of one particle, C is a constant, see [10].

Let us make the substitution to dimensionless variables $\vec{q}$ and $\vec{q}'$:

$$\vec{p} = \sqrt{mT}\,\vec{q}\,,\ \ \vec{p}' = \sqrt{mT}\,\vec{q}'\,. \tag{4}$$

On the basis of (3) and (4) we obtain

$$H_{js} = C\left(mT\right)^{7/2}\int d^3q d^3q' \exp\left(-\left(q^2+q'^2\right)/2\right) D_{nk,qq'} B^{(j)}_{nlm,qq'} B^{(s)}_{klm,qq'} \tag{5}$$

where

$$\begin{aligned}
B^{(j)}_{nlm,qq'} &= \frac{\partial\left(q_l q_m - \delta_{ml} q^2/3\right) S_j^{5/2}\left(q^2/2\right)}{\partial q_n} - \frac{\partial\left(q'_l q'_m - \delta_{ml} q'^2/3\right) S_j^{5/2}\left(q'^2/2\right)}{\partial q'_n} = \\
&= \left(\delta_{ln} q_m + \delta_{mn} q_l - \frac{2}{3}\delta_{ml} q_n\right) S_j^{5/2}\left(\frac{q^2}{2}\right) + \frac{\partial S_j^{5/2}\left(q^2/2\right)}{\partial\left(q^2/2\right)} q_n \left(q_l q_m - \frac{1}{3}\delta_{ml} q^2\right) - \\
&- \left(\delta_{ln} q'_m + \delta_{mn} q'_l - \frac{2}{3}\delta_{ml} q'_n\right) S_j^{5/2}\left(\frac{q'^2}{2}\right) - \left(q'_l q'_m - \frac{1}{3}\delta_{ml} q'^2\right) q'_n \frac{\partial S_j^{5/2}\left(q'^2/2\right)}{\partial\left(q'^2/2\right)}\,.
\end{aligned} \tag{6}$$

By a straightforward calculation on the basis of (6) it may be shown that

$$\begin{aligned}
B^{(j)}_{nlm,qq'} B^{(s)}_{klm,qq'} &= 2\left(\delta_{kn} q^2 + \frac{1}{3} q_n q_k\right) S_{jq} S_{sq} + 2\left(\delta_{kn} q'^2 + \frac{1}{3} q'_n q'_k\right) S_{jq'} S_{sq'} - \\
&- 2\left(\delta_{nk}\left(qq'\right) + q_k q'_n - \frac{2}{3} q_n q'_k\right) S^{5/2}_{jq} S^{5/2}_{sq'} - 2\left(\delta_{kn}\left(qq'\right) + q'_k q_n - \frac{2}{3} q'_n q_k\right) S^{5/2}_{jq'} S^{5/2}_{sq} + \\
&+ \frac{2}{3} q^4 q_n q_k \Delta_{jq} \Delta_{sq} + \frac{2}{3} q'^4 q'_n q'_k \Delta_{jq'} \Delta_{sq'} - \\
&- \left(\left(qq'\right)^2 - \frac{1}{3} q'^2 q^2\right)\left(q'_n q_k \Delta_{jq'} \Delta_{sq} + q_n q'_k \Delta_{jq} \Delta_{sq'}\right) + \\
&+ \frac{4}{3} q_n q_k q^2 \left(\Delta_{jq} S_{sq} + \Delta_{sq} S_{jq}\right) + \frac{4}{3} q'^2 q'_k q'_n \left(\Delta_{jq'} S_{sq'} + \Delta_{sq'} S_{jq'}\right) - \\
&- 2\left(q'_k q'_n \left(qq'\right) - \frac{1}{3} q'^2 q'_n q_k\right) \Delta_{jq'} S_{sq} - 2\left(q_k q_n \left(qq'\right) - \frac{1}{3} q^2 q'_n q_k\right) S_{jq'} \Delta_{sq} - \\
&- 2\left(q_n q_k \left(qq'\right) - \frac{1}{3} q^2 q_n q'_k\right) \Delta_{jq} S_{sq'} - 2\left(q'_n q'_k \left(qq'\right) - \frac{1}{3} q'^2 q_n q'_k\right) S_{jq} S_{sq'}
\end{aligned} \tag{7}$$

where

$$\begin{gathered}
S_{jq} = S_j^{5/2}\left(\frac{q^2}{2}\right),\ \Delta_{jq} = \frac{\partial S_j^{5/2}\left(q^2/2\right)}{\partial\left(q^2/2\right)}\,, \\
\left(qq'\right) = \left(\vec{q},\vec{q}'\right) = q_l q'_l\,,\ q^2 = \left(\vec{q},\vec{q}\right) = q_l q_l\,,\ q'^2 = \left(\vec{q}',\vec{q}'\right) = q'_l q'_l\,.
\end{gathered} \tag{8}$$

Then the following transform is made:

$$\vec{x} = \vec{q} + \vec{q}', \ \vec{y} = \vec{q} - \vec{q}', \ \vec{x} = \left(x\sin\theta_x \cos\varphi_x, x\sin\theta_x \sin\varphi_x, x\cos\theta_x \right), \quad \vec{y} = \left(y\sin\theta_y \cos\varphi_y, y\sin\theta_y \sin\varphi_y, y\cos\theta_y \right). \tag{9}$$

On the basis of (9) we obtain

$$\vec{q} = \frac{\vec{x}+\vec{y}}{2}, \ \vec{q}' = \frac{\vec{x}-\vec{y}}{2}, \ d^3q d^3q' = \frac{1}{8} d^3x d^3y, \tag{10}$$

and

$$q^2 = \frac{x^2 + 2xy\Omega + y^2}{4}, \ q'^2 = \frac{x^2 - 2xy\Omega + y^2}{4},$$
$$\left(q,q'\right) = \frac{x^2 - y^2}{4}, \ D_{nk,qq'}\delta_{nk} = \frac{2}{y}, \tag{11}$$
$$D_{nk,qq'}q_n q_k = D_{nk,qq'}q'_n q'_k = D_{nk,qq'}q_n q'_k = D_{nk,qq'}q'_n q_k = \frac{x^2\left(1-\Omega^2\right)}{4y},$$

see [10], where

$$\Omega = \sin\theta_x \cos\varphi_x \sin\theta_y \cos\varphi_y + \sin\theta_x \sin\varphi_x \sin\theta_y \sin\varphi_y + \cos\theta_x \cos\theta_y . \tag{12}$$

On the basis of (11), (7) and (8) by a straightforward calculation one can conclude that

$$\begin{aligned}
D_{nk,qq'}B^{(j)}_{nlm,qq'}B^{(s)}_{klm,qq'} = f\left(x,y,\Omega\right) = &\left(\frac{x^2 + 2xy\Omega + y^2}{y} + \frac{x^2 - x^2\Omega^2}{6y} \right) S_{jq}S_{sq} + \\
&+\left(\frac{x^2 - 2xy\Omega + y^2}{y} + \frac{x^2 - x^2\Omega^2}{6y} \right) S_{jq'}S_{sq'} - \\
&-\left(\frac{x^2 - y^2}{y} + \frac{x^2 - x^2\Omega^2}{6y} \right) \left(S_{jq}S_{sq'} + S_{jq'}S_{sq} \right) + \\
&+\frac{x^2 - x^2\Omega^2}{6y} \left(\left(\frac{x^2 + 2xy\Omega + y^2}{4} \right)^2 \Delta_{jq}\Delta_{sq} + \left(\frac{x^2 - 2xy\Omega + y^2}{4} \right)^2 \Delta_{jq'}\Delta_{sq'} \right) - \\
&-\frac{x^2 - x^2\Omega^2}{4y} \left(\left(\frac{x^2 - y^2}{4} \right)^2 - \frac{1}{3} \frac{\left(x^2+y^2\right)^2 - 4x^2y^2\Omega^2}{16} \right) \left(\Delta_{jq'}\Delta_{sq} + \Delta_{jq}\Delta_{sq'} \right) + \\
&+\frac{x^2 - x^2\Omega^2}{3y} \frac{x^2 + 2xy\Omega + y^2}{4} \left(\Delta_{jq}S_{sq} + \Delta_{sq}S_{jq} \right) + \\
&+\frac{x^2 - x^2\Omega^2}{3y} \frac{x^2 - 2xy\Omega + y^2}{4} \left(\Delta_{jq'}S_{sq'} + \Delta_{sq'}S_{jq'} \right) - \\
&-\frac{x^2 - x^2\Omega^2}{2y} \left(\frac{x^2 - y^2}{4} - \frac{1}{3} \frac{x^2 - 2xy\Omega + y^2}{4} \right) \left(\Delta_{jq'}S_{sq} + S_{jq}\Delta_{sq'} \right) -
\end{aligned} \tag{13}$$

$$-\frac{x^2-x^2\Omega^2}{2y}\left(\frac{x^2-y^2}{4}-\frac{1}{3}\frac{x^2+2xy\Omega+y^2}{4}\right)\left(S_{jq'}\Delta_{sq}+\Delta_{jq}S_{sq'}\right)$$

where

$$S_{jq}=S_j^{5/2}\left(\frac{q^2}{2}\right)=S_j^{5/2}\left(\frac{x^2+2xy\Omega+y^2}{8}\right),\ \Delta_{jq}=\left.\frac{\partial S_j^{5/2}(z)}{\partial z}\right|_{z=\frac{x^2+2xy\Omega+y^2}{8}},$$
$$S_{jq'}=S_j^{5/2}\left(\frac{q'^2}{2}\right)=S_j^{5/2}\left(\frac{x^2-2xy\Omega+y^2}{8}\right),\ \Delta_{jq'}=\left.\frac{\partial S_j^{5/2}(z)}{\partial z}\right|_{z=\frac{x^2-2xy\Omega+y^2}{8}}. \tag{14}$$

On the basis of (5), (10), (13) it is seen that the integral brackets under consideration are as follows:

$$H_{js}=\frac{C(mT)^{7/2}}{8}\int d^3x d^3y e^{-\frac{x^2+y^2}{4}} f\left(x,y,\Omega\left(\theta_x,\theta_y,\varphi_x,\varphi_y\right)\right)=$$
$$=\frac{C(mT)^{7/2}}{8}\cdot\int_0^\infty dx\int_0^\infty dy\int_0^\pi d\theta_x\int_0^\pi d\theta_y\int_0^{2\pi} d\varphi_x\int_0^{2\pi} d\varphi_y g\left(x,y,\Omega\left(\theta_x,\theta_y,\varphi_x,\varphi_y\right)\right), \tag{15}$$

where

$$g\left(x,y,\Omega\left(\theta_x,\theta_y,\varphi_x,\varphi_y\right)\right)=x^2y^2\sin\theta_x\sin\theta_y e^{-\frac{x^2+y^2}{4}} f\left(x,y,\Omega\left(\theta_x,\theta_y,\varphi_x,\varphi_y\right)\right), \tag{16}$$

the explicit expression for $\Omega\left(\theta_x,\theta_y,\varphi_x,\varphi_y\right)$ is given in (12). It can be shown that the expression $x^2y^2f(x,y,\Omega)$ is a sum of some terms which have the form $ax^by^c\Omega^d$ where a is a constant, b is an even integer not less than 2, c is an odd integer not less than 3 and d is an even integer not less than 0. In [10] it is shown that for integer non-negative n

$$\int_0^\pi d\theta_x\int_0^\pi d\theta_y\int_0^{2\pi} d\varphi_x\int_0^{2\pi} d\varphi_y \sin\theta_x\sin\theta_y\Omega^{2n}\left(\theta_x,\theta_y,\varphi_x,\varphi_y\right)=$$
$$=\frac{16\pi^2}{2n+1}=16\pi^2\int_0^1\Omega^{2n}d\Omega \tag{17}$$

where in the latter integral Ω is considered as a one-dimensional variable. So, the integral brackets may be rewritten as follows

$$H_{js}=2\pi^2C(mT)^{7/2}\int_0^\infty dx\int_0^\infty dy\int_0^1 d\Omega x^2y^2e^{-\frac{x^2+y^2}{4}} f(x,y,\Omega) \tag{18}$$

where Ω is considered as a one-dimensional variable. The further calculation of (18) in general case is too cumbersome, and it is provided in the Wolfram Mathematica package. The explicit analytical results for H_{js} are given in Appendix A.

Conclusions. The exact analytical results for the velocity integral brackets are calculated up to the thirteen-polynomial approximation. The result for H_{00} coincides with [4], the other results are new ones. The obtained integral brackets are important for further numerical investigation of the convergence of the results for the viscosity of the one-component system with weak interaction with increasing of the number of polynomials.

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Appendix A. **Explicit results for *H_{js}*.** Let us denote

$$H_{js} = C\pi^{5/2}\left(mT\right)^{7/2} Y_{j,s} \big/ 8 . \qquad (A.1)$$

Due to the symmetry of integral brackets, we have

$$Y_{j,s} = Y_{s,j} . \qquad (A.2)$$

The following explicit results for Y_{js} are obtained:

$$\begin{gathered} Y_{0,0} = 1024 \, , Y_{1,0} = 768 , \; Y_{1,1} = \frac{13120}{3} , \; Y_{2,0} = 480 \, , \; Y_{2,1} = 3912 \, , \\ Y_{2,2} = 11889 \, , \; Y_{3,0} = 280 , \; Y_{3,1} = 2870 \, , \; Y_{3,2} = \frac{45129}{4} , \; Y_{3,3} = \frac{1220437}{48} \, , \\ Y_{4,0} = \frac{315}{2} \, , \; Y_{4,1} = \frac{46235}{24} , \; Y_{4,2} = \frac{573045}{64} \, , \; Y_{4,3} = \frac{6323017}{256} , \\ Y_{4,4} = \frac{575512075}{12288} , \; Y_{5,0} = \frac{693}{8} \, , \; Y_{5,1} = \frac{39249}{32} \, , \; Y_{5,2} = \frac{1667295}{256} \, , \\ Y_{5,3} = \frac{20967525}{1024} \, , \; Y_{5,4} = \frac{753446175}{16384} , \; Y_{5,5} = \frac{5109697839}{65536} \, , \; Y_{6,0} = \frac{3003}{64} , \end{gathered} \qquad (A.3)$$

$$Y_{6,1}=\frac{192885}{256},\ Y_{6,2}=\frac{9180549}{2048},\ Y_{6,3}=\frac{384608735}{24576},\ Y_{6,4}=\frac{5136713305}{131072},$$

$$Y_{6,5}=\frac{40354951347}{524288},\ Y_{6,6}=\frac{1518412747195}{12582912},\ Y_{7,0}=\frac{6435}{256},\ Y_{7,1}=\frac{461747}{1024},$$

$$Y_{7,2}=\frac{24298857}{8192},\ Y_{7,3}=\frac{371516607}{32768},\ Y_{7,4}=\frac{48725240795}{1572864},$$

$$Y_{7,5}=\frac{140214444045}{2097152},\ Y_{7,6}=\frac{2004553960213}{16777216},\ Y_{7,7}=\frac{35594203209637}{201326592},$$

$$Y_{8,0}=\frac{109395}{8192},\ Y_{8,1}=\frac{8667945}{32768},\ Y_{8,2}=\frac{499308381}{262144},\ Y_{8,3}=\frac{8280211401}{1048576},$$

$$Y_{8,4}=\frac{390512910753}{16777216},\ Y_{8,5}=\frac{3635029390335}{67108864},\ Y_{8,6}=\frac{56509668162465}{536870912},$$

$$Y_{8,7}=\frac{376553553236937}{2147483648},\ Y_{8,8}=\frac{17055696189814761}{68719476736},\ Y_{9,0}=\frac{230945}{32768},$$

$$Y_{9,1}=\frac{20019285}{131072},\ Y_{9,2}=\frac{1252343235}{1048576},\ Y_{9,3}=\frac{22367962617}{4194304},$$

$$Y_{9,4}=\frac{1129739588835}{67108864},\ Y_{9,5}=\frac{11229214999275}{268435456},$$

$$Y_{9,6}=\frac{559841408135765}{6442450944},\ Y_{9,7}=\frac{1341443094257665}{8589934592},$$

$$Y_{9,8}=\frac{67729866100480707}{274877906944},\ Y_{9,9}=\frac{1110631375234799365}{3298534883328},$$

$$Y_{10,0}=\frac{969969}{262144},\ Y_{10,1}=\frac{273854581}{3145728},\ Y_{10,2}=\frac{6161089935}{8388608},$$

$$Y_{10,3}=\frac{117856931335}{33554432},\ Y_{10,4}=\frac{19021888785185}{1610612736},$$

$$Y_{10,5}=\frac{668889090399249}{2147483648},\ Y_{10,6}=\frac{1178107821992955}{17179869184},$$

$$Y_{10,7}=\frac{26981897385329845}{206158430208},\ Y_{10,8}=\frac{486791876649554595}{2199023255552},$$

$$Y_{10,9}=\frac{2942063302892776685}{8796093022208},\ Y_{10,10}=\frac{93769137202885050451}{211106232532992},$$

$$Y_{11,0}=\frac{2028117}{1048576},\ Y_{11,1}=\frac{205897965}{4194304},\ Y_{11,2}=\frac{14910854751}{33554432},$$

$$Y_{11,3}=\frac{304065467745}{134217728},\ Y_{11,4}=\frac{17350825762335}{2147483648},$$

$$Y_{11,5}=\frac{193416418687635}{8589934592},\ Y_{11,6}=\frac{3591354358261395}{68719476736},$$

$$Y_{11,7}=\frac{28890564218427177}{274877906944},\ Y_{11,8}=\frac{1652117128117741215}{8796093022208},$$

$$Y_{11,9}=\frac{10648816201945807455}{35184372088832},\ Y_{11,10}=\frac{124243744340141169105}{281474976710656},$$

$$Y_{11,11}=\frac{644539772995200419895}{1125899906842624},\ Y_{12,0}=\frac{16900975}{16777216},$$

$$Y_{12,1}=\frac{1840854197}{67108864}\ ,\ Y_{12,2}=\frac{142367023617}{536870912}\ ,\ Y_{12,3}=\frac{9247595464991}{6442450944}\ ,$$

$$Y_{12,4}=\frac{185907178821845}{34359738368}\ ,\ Y_{12,5}=\frac{2182405790407635}{137438953472}\ ,$$

$$Y_{12,6}=\frac{127684551050232319}{3298534883328}\ ,\ Y_{12,7}=\frac{359073588842935989}{4398046511104}\ ,$$

$$Y_{12,8}=\frac{21538258527186410877}{140737488355328}\ ,\ Y_{12,9}=\frac{438188089222250610005}{1688849860263936}\ ,$$

$$Y_{12,10}=\frac{1809687195463548779135}{4503599627370496}\ ,\ Y_{12,11}=\frac{10250556810713117245587}{18014398509481984}\ ,$$

$$Y_{12,12}=\frac{625526736346053210664867}{864691128455135232}\ ,\ Y_{13,0}=\frac{35102025}{67108864}\ ,$$

$$Y_{13,1}=\frac{12248006575}{805306368}\ ,\ Y_{13,2}=\frac{335806720347}{2147483648}\ ,\ Y_{13,3}=\frac{7693283024777}{8589934592}\ ,$$

$$Y_{13,4}=\frac{1466672846806673}{412316860416}\ ,\ Y_{13,5}=\frac{6026823411518235}{549755813888}\ ,$$

$$Y_{13,6}=\frac{123095520994962815}{4398046511104}\ ,\ Y_{13,7}=\frac{3256684072229294707}{52776558133248}\ ,$$

$$Y_{13,8}=\frac{68004336417310283067}{562949953421312}\ ,\ Y_{13,9}=\frac{481932498552841693959}{2251799813685248}\ ,$$

$$Y_{13,10}=\frac{18781211615862755037055}{54043195528445952}\ ,$$

$$Y_{13,11}=\frac{37519533823987500972495}{72057594037927936}\ ,$$

$$Y_{13,12}=\frac{829137898242603453004639}{1152921504606846976}\ ,$$

$$Y_{13,13}=\frac{12435770928858947379094117}{13835058055282163712}\ .$$

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